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## Mirror Symmetry

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### Abstract

We prove mirror symmetry for supersymmetric sigma models on Kahler manifolds in  $1+1$  dimensions. The proof involves establishing the equivalence of the gauged linear sigma model, embedded in a theory with an enlarged gauge symmetry, with a Landau-Ginzburg theory of Toda type. Standard  $R \rightarrow 1/R$  duality and dynamical generation of superpotential by vortices are crucial in the derivation. This provides not only a proof of mirror symmetry in the case of (local and global) Calabi-Yau manifolds, but also for sigma models on manifolds with positive first Chern class, including deformations of the action by holomorphic isometries.

# 1 Introduction

One of the most beautiful symmetries in string theory is the radius inversion symmetry  $R \rightarrow 1/R$  of a circle, known as T-duality [1]. This is the symmetry which exchanges winding modes on a circle with momentum modes on the dual circle. This symmetry has been the underlying motivation for many of the subsequent dualities discovered in string theory and in quantum field theories. In particular, over a decade ago, it was conjectured in [2, 3] that a similar duality might exist in the context of string propagation on Calabi-Yau manifolds, where the role of the complex deformations on one manifold get exchanged with the Kahler deformations on the dual manifold. The pairs of manifolds satisfying this symmetry are known as mirror pairs, and this duality is also called mirror symmetry.

There has been a lot of progress since the original formulation of this conjecture, in its support. In particular many examples of this phenomenon were found [4, 5]. The intermediate step in the derivation for this class of examples involved the construction of conformal field theory for certain Calabi-Yau's [6] and their identification with certain Landau-Ginzburg models [7–9]. This connection was further elucidated in [10] where it was shown that the linear sigma model is a powerful tool in the study of strings propagating on a Kahler manifold.

It was shown in [11] how mirror symmetry can be used very effectively to gain insight into non-perturbative effects involving worldsheet instantons. Roughly speaking this amounts to counting the number of holomorphic curves in a Calabi-Yau manifold. This made the subject also interesting for algebraic geometers in the context of enumerative geometry. Motivated by the existing examples some general class of mirror pairs were formulated by mathematicians using toric geometry [12]. Moreover a program to prove the rational curve counting formula, predicted by mirror symmetry, from the view point of localization and virtual fundamental cycles was initiated in [13–15] and was pushed to completion in [16–19]. For reviews of various aspects of mirror symmetry see [20]; for mathematical aspects of mirror symmetry see the excellent book [21].

The question of a proof of mirror symmetry and its relation with T-duality, which was its original motivation, was further pursued in [22] where it was shown that for certain toroidal orbifold models mirror symmetry reduces to T-duality. More generally, by following the prediction of the map of D-branes under mirror symmetry, it was argued in [23] that mirror symmetry should reduce to T-duality in a more general context. Furthermore, it was shown in [24], how the general suggestion for construction of mirror pairs proposed using toric geometry [12] can be intuitively related to T-duality.

Mirror symmetry has also been extended from the case of Calabi-Yau sigma models to more general cases. On the one hand there are proposals as to what the mirror theories are in the case of certain sigma models with positive first Chern class [25–29, 16]. On the other hand there are proposals for what the mirror of non-compact Calabi-Yau manifolds are [30–32].

The aim of this paper is to present a proof of mirror symmetry for all cases proposed thus far. The proof depends crucially on establishing a dual description of  $(2, 2)$  supersymmetric gauge theories in  $1+1$  dimensions. The dual theory is found using the idea analogous to Polyakov’s model of confinement in quantum electrodynamics in  $2 + 1$  dimensions [33]. He considered a  $U(1)$  gauge theory which includes magnetic monopoles playing the role of instantons.  $U(1)$  Maxwell theory of gauge coupling constant  $e$  in  $2 + 1$  dimensions is dual to the theory of a periodic scalar field  $\sigma \equiv \sigma + 2\pi$  with the Lagrangian  $e^2|d\sigma|^2$ . The gas of instantons and anti-instantons with a long range interaction between them generates a potential term

$$U(\sigma) = \mu^3 \cos(\sigma) \tag{1.1}$$

in the effective Lagrangian in terms of the dual variable  $\sigma$ , where  $\mu$  is the mass scale determined by  $e$  and the monopole size. One sees from this that a mass gap is generated and that an electric flux is confined into a thin tube. We note that the description in terms of the dual variable  $\sigma$  was essential in this argument.

In supersymmetric field theories, instanton computation can be used to obtain exact results for some important physical quantities. For some of the striking examples, see [34–39]. Among these, [34] and [39] treat supersymmetric gauge theories in  $2 + 1$  dimensions and the effective theory is described in terms of the dual variable as in [33]. Also, in [38], duality between vector and vector in  $3 + 1$  dimensions was used to solve the problem in an essential way.

We apply an analogous idea to study the long distance behaviour of  $(2, 2)$  gauge theories, making use of instantons which are vortices [40] in this case. We dualize the phase of the charged fields in the sense of  $R \rightarrow 1/R$  duality and describe the low energy effective theory in terms of the dual variables. We will see that a superpotential is dynamically generated by the instanton effect, as in [33–35], and we can exactly determine the (twisted) F-term part of the effective Lagrangian. To be specific, let us consider a  $(2, 2)$  supersymmetric  $U(1)$  gauge theory with  $N$  chiral multiplets of charge  $Q_i$  ( $i = 1, \dots, N$ ). In addition to the gauge coupling, the theory has two parameters: Fayet-Iliopoulos and Theta parameters. They are combined into a single complex parameter  $t$  and appear in the twisted superpotential as  $-t\Sigma$  where  $\Sigma$  is the twisted chiral field which is the field strength of

the gauge multiplet (and includes the scalar, the gaugino and the field strength). Each charged chiral field is sent by the duality on its phase to a twisted chiral field  $Y_i$  which is a neutral periodic variable  $Y_i \equiv Y_i + 2\pi i$  that couples to the field strength as a dynamical Theta angle  $Q_i Y_i \Sigma$ . The exact twisted superpotential we will find is given by

$$\widetilde{W} = \Sigma \left( \sum_{i=1}^N Q_i Y_i - t \right) + \mu \sum_{i=1}^N e^{-Y_i}, \quad (1.2)$$

where  $\mu$  is a scale parameter. The term proportional to  $\Sigma$  is the one that appears already at the dualization process. The exponentials of  $Y_i$ 's are the ones that are generated by instanton effect. When  $\sum_i Q_i \neq 0$ ,  $\mu$  is a scale required to renormalize the FI parameter  $t$ . In this case, a combination of  $t$  and  $\mu$  is a fake and only one dimensionful parameter  $\Lambda = \mu e^{-t/\sum_i Q_i}$  is the real parameter of the theory. This is the standard dimensional transmutation. In the case where  $\sum_i Q_i = 0$ ,  $t$  is the dimensionless parameter of the theory and  $\mu$  is a fake as it can be absorbed by a field redefinition.

Using the connection between  $U(1)$  gauge theories with matter and sigma models on Kahler manifolds [10] we then relate the above result to the statement of mirror symmetry<sup>1</sup>. In particular we find that the mirror to a sigma model is a Landau-Ginzburg model. In the case of Calabi-Yau manifolds this can also be related to the sigma model on another Calabi-Yau manifold by the equivalence of sigma models and Landau-Ginzburg models. In the case of manifolds with non-zero first Chern class, however, this is not possible (we consider only manifolds with non-negative first Chern class since otherwise the sigma model would not be well-defined): the axial  $U(1)$  R-symmetry is broken by an anomaly and therefore the vector  $U(1)$  R-symmetry of the mirror theory must be broken by an inhomogeneous superpotential. Likewise, since the vector  $U(1)$  R-symmetry of the original non-linear sigma model is an exact symmetry, the mirror manifold (on which the Landau-Ginzburg superpotential is defined) must always be Calabi-Yau so that the axial R-symmetry is unbroken.

A typical example of manifolds of positive first Chern class is  $\mathbf{CP}^1$ . It has been observed that the supersymmetric  $\mathbf{CP}^1$  sigma model is mirror to the  $N = 2$  sine-Gordon theory which is a sigma model on a cylinder  $\mathbf{C}^\times$  with a sine-Gordon superpotential [25–29]. The two theories have  $U(1) \times \mathbf{Z}_4$  vector-axial (or axial-vector) R-symmetries. Both have two massive vacua which spontaneously breaks  $\mathbf{Z}_4$  to  $\mathbf{Z}_2$ . Moreover, soliton spectrum and the scattering matrix have been observed to agree [25]. Actually, this mirror symmetry is the first non-trivial one that can be derived by our method. The linear sigma model for  $\mathbf{CP}^1$  is a  $U(1)$  gauge theory with two chiral multiplets of charge  $Q_1 = Q_2 = 1$ . By

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<sup>1</sup>The idea to use the gauged linear sigma model to derive mirror symmetry was also considered in [41].

integrating out the  $\Sigma$  field from (1.2), we obtain the constraint  $Y_1 + Y_2 = t$  which can be solved by  $Y_1 = Y + t/2, Y_2 = -Y + t/2$ . Then, the superpotential (1.2) takes the form

$$\widetilde{W}(Y) = 2\Lambda \cosh(Y), \quad (1.3)$$

which is nothing but the sine-Gordon potential! In fact, the mirror symmetry of other models, including the conformal field theories based on Calabi-Yau sigma models, can be derived in a uniform way as a natural generalization of this example. Furthermore, the mirror theory provides an effective way to classify the vacua of the theory and to identify where they flow to in the infra-red limit. For example, for a degree  $d$  hypersurface on  $\mathbf{CP}^{N-1}$  of size  $t$ , we find the mirror to be the orbifold of the Landau-Ginzburg model with the superpotential

$$W = X_1^d + \cdots + X_N^d + e^{t/d} X_1 \cdots X_N \quad (1.4)$$

by the group  $(\mathbf{Z}_d)^{N-1}$  which acts on the individual fields  $X_i$  by multiplication by  $d$ -th roots of unity, in such a way that  $X_1 \cdots X_N$  is invariant. Note that the special case  $d = N$  gives one the proposed mirror of sigma models on Calabi-Yau hypersurfaces (Greene-Plesser construction [4]). In the case with  $d < N$ , the hypersurface has a positive first Chern class and the sigma model is asymptotic free [42]. The mirror theory given above shows that there are  $N - d$  massive vacua at non-zero  $X_i$ 's and another vacuum at  $X_i = 0$ . For  $d > 2$ , the vacuum at  $X_i = 0$  flows to a non-trivial fixed point described by the LG orbifold with the same group  $(\mathbf{Z}_d)^{N-1}$  and the superpotential (1.4) with the last term being dropped as it is irrelevant at low energies. In this example, the original linear sigma model actually leads to the description of the low energy theory in terms of another LG orbifold [10, 43], with the same superpotential but with a different group — a single  $\mathbf{Z}_d$  acting on  $X_i$ 's uniformly. In fact, our result reproduces the mirror symmetry of LG orbifolds [44]. In general, however, as the  $\mathbf{CP}^1$  example shows, our method provide information which is hardly available in the original model.

The organization of this paper is as follows: In section 2 we review certain aspects of mirror symmetry with emphasis on its interplay with supersymmetry. We present the  $N = 2$  supersymmetric version of T-duality which plays a crucial role for us later in the paper. In section 3 we consider dynamical aspects of  $N = 2$  gauge theories and establish their equivalence with the above mentioned LG theories. In section 4 we review aspects of linear sigma model (i.e. gauge theory/sigma model connection). In section 5 we use the results in section 3 and present a proof of mirror symmetry. Also in this section we elaborate on what we mean by “proving” mirror symmetry. In section 6 we discuss some aspects of D-branes in the context of LG theories. This elucidates the relation between mirror symmetry for local (non-compact) and global (i.e. compact)

sigma models which is discussed in section 7. Also the mirror of complete intersections in toric varieties are discussed there. In section 8 we discuss some possible directions for future work. In appendix A we present a conjectured generalization of our results to the case of complete intersections in Grassmannians and flag varieties. In appendix B some aspects of supersymmetry transformations needed in this paper are summarized.

## 2 Mirror Symmetry Of $N = 2$ Theories In Two Dimensions

In this section, we review some basic facts and fix notations on  $(2, 2)$  supersymmetric field theories in 1+1 dimensions. We also define the notion of mirror symmetry and present some examples. In particular, we describe in detail the standard  $R \rightarrow 1/R$  duality and show how it can be viewed as mirror symmetry in the case of complex torus.

### 2.1 Supersymmetry

#### $(2, 2)$ Supersymmetry Algebra

$(2, 2)$  supersymmetry algebra is generated by four supercharges  $Q_{\pm}, \overline{Q}_{\pm}$ , space-time translations  $P, H$  and rotation  $M$ , and the generators  $F_V$  and  $F_A$  of two R-symmetries  $U(1)_V$  and  $U(1)_A$ . These obey the following (anti-)commutation relations:

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \quad (2.1)$$

$$\{Q_{\pm}, \overline{Q}_{\pm}\} = 2(H \mp P), \quad (2.2)$$

$$\{\overline{Q}_+, \overline{Q}_-\} = 2Z, \quad \{Q_+, Q_-\} = 2Z^*, \quad (2.3)$$

$$\{Q_-, \overline{Q}_+\} = 2\tilde{Z}, \quad \{Q_+, \overline{Q}_-\} = 2\tilde{Z}^*, \quad (2.4)$$

$$[M, Q_{\pm}] = \mp Q_{\pm}, \quad [M, \overline{Q}_{\pm}] = \mp \overline{Q}_{\pm}, \quad (2.5)$$

$$[F_V, Q_{\pm}] = -Q_{\pm}, \quad [F_V, \overline{Q}_{\pm}] = \overline{Q}_{\pm}, \quad (2.6)$$

$$[F_A, Q_{\pm}] = \mp Q_{\pm}, \quad [F_A, \overline{Q}_{\pm}] = \pm \overline{Q}_{\pm}. \quad (2.7)$$

The hermiticity of the generators is dictated by

$$Q_{\pm}^{\dagger} = \overline{Q}_{\pm}. \quad (2.8)$$

In the above expressions,  $Z$  and  $\tilde{Z}$  are central charges. The algebra with  $Z = \tilde{Z} = 0$  can be obtained by dimensional reduction of the four-dimensional  $N = 1$  supersymmetry algebra.  $U(1)_V$  comes from the R-symmetry in four dimensions and  $U(1)_A$  is the rotation along the reduced directions.

It is not always the case that the two  $U(1)$  R-symmetries are the symmetry of a given theory, except in *superconformal* field theories where they both *must* be the symmetry. In the class of theories we will consider in this paper, at least one of them is a symmetry and the other may or may not be broken to a discrete subgroup.

A central charge can be non-zero if there is a soliton that interpolates different vacua and/or if the theory has a continuous abelian symmetry. As the name suggests, it must commute with the R-symmetry generators as well. In particular,  $Z$  ( $\tilde{Z}$ ) must always be zero in a theory where  $U(1)_V$  ( $U(1)_A$ ) is unbroken. In superconformal field theory, both must be vanishing.

### *Superfields and Supersymmetric Lagrangians*

Fields in a supermultiplet can be combined into a single function, a *superfield*, of the superspace coordinates  $x^0, x^1, \theta^\pm, \bar{\theta}^\pm$ . Supersymmetry generators  $Q_\pm, \bar{Q}_\pm$  act on the superfields as the derivatives

$$Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\bar{\theta}^\pm \left( \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right), \quad \bar{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - i\theta^\pm \left( \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right). \quad (2.9)$$

These commute with another set of derivatives

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - i\bar{\theta}^\pm \left( \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right), \quad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i\theta^\pm \left( \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right). \quad (2.10)$$

R-symmetries act on a superfield  $\mathcal{F}(x, \theta^\pm, \bar{\theta}^\pm)$  as

$$e^{i\alpha F_V} \mathcal{F}(x, \theta^\pm, \bar{\theta}^\pm) = e^{iq_V \alpha} \mathcal{F}(x, e^{-i\alpha} \theta^\pm, e^{i\alpha} \bar{\theta}^\pm), \quad (2.11)$$

$$e^{i\alpha F_A} \mathcal{F}(x, \theta^\pm, \bar{\theta}^\pm) = e^{iq_A \alpha} \mathcal{F}(x, e^{\mp i\alpha} \theta^\pm, e^{\pm i\alpha} \bar{\theta}^\pm), \quad (2.12)$$

where  $q_V$  and  $q_A$  are the vector and axial R-charges of  $\mathcal{F}$ .

The basic representations of the supersymmetry algebra are chiral and twisted chiral multiplets which both consist of a complex scalar and a Dirac fermion. These are represented by chiral (or *cc*) superfield and twisted chiral (or *ac*) superfield respectively. A chiral superfield  $\Phi$  satisfies

$$\bar{D}_\pm \Phi = 0, \quad (2.13)$$

and can be expanded as

$$\Phi = \phi + \sqrt{2}\theta^+ \psi_+ + \sqrt{2}\theta^- \psi_- + 2\theta^+ \theta^- F + \dots, \quad (2.14)$$

where  $F$  is a complex auxiliary field and  $+\dots$  involves only the derivatives of  $\phi, \psi_{\pm}$ . The hermitian conjugate of  $\Phi$  is an anti-chiral (or  $aa$ ) superfield  $D_{\pm}\bar{\Phi} = 0$ . A twisted chiral superfield  $Y$  satisfies [45]

$$\bar{D}_+Y = D_-Y = 0, \quad (2.15)$$

and can be expanded as

$$Y = y + \sqrt{2}\theta^+\bar{\chi}_+ + \sqrt{2}\bar{\theta}^-\chi_- + 2\theta^+\bar{\theta}^-G + \dots \quad (2.16)$$

where  $G$  is a complex auxiliary field and  $+\dots$  involves only the derivatives of the component fields. The hermitian conjugate of  $Y$  is a twisted anti-chiral (or  $ca$ ) superfield  $D_+\bar{Y} = \bar{D}_-\bar{Y} = 0$ .

We also introduce a vector multiplet. It consists of a vector field  $v_{\mu}$ , Dirac fermions  $\lambda_{\pm}$ ,  $\bar{\lambda}_{\pm}$  which are conjugate to each other, and a complex scalar  $\sigma$  in the adjoint representation of the gauge group. It is represented in a vector superfield  $V$  which is expanded (in the Wess-Zumino gauge) as

$$\begin{aligned} V = & \theta^-\bar{\theta}^-(v_0 - v_1) + \theta^+\bar{\theta}^+(v_0 + v_1) - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma} \\ & + \sqrt{2}i\theta^-\theta^+(\bar{\theta}^-\bar{\lambda}_- + \bar{\theta}^+\bar{\lambda}_+) + \sqrt{2}i\bar{\theta}^+\bar{\theta}^-(\theta^-\lambda_- + \theta^+\lambda_+) + 2\theta^-\theta^+\bar{\theta}^+\bar{\theta}^-D \end{aligned} \quad (2.17)$$

where  $D$  is a real auxiliary field. Using the gauge covariant derivatives  $\mathcal{D}_{\pm} = e^{-V}D_{\pm}e^V$ ,  $\bar{\mathcal{D}}_{\pm} = e^V\bar{D}_{\pm}e^{-V}$ , we can define the field strength as

$$\begin{aligned} \Sigma &= \frac{1}{2}\{\bar{\mathcal{D}}_+, \mathcal{D}_-\} \\ &= \sigma + i\sqrt{2}\theta^+\bar{\lambda}_+ - i\sqrt{2}\bar{\theta}^-\lambda_- + 2\theta^+\bar{\theta}^-(D - iF_{01}) + \dots, \end{aligned} \quad (2.18)$$

where  $F_{01}$  is the curvature of  $v_{\mu}$ . This is a twisted chiral (covariant) superfield  $\bar{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$ . The supersymmetry transformation of the component fields in this gauge (based on [46, 47, 10]) is recorded for convenience in Appendix B.

Supersymmetric Lagrangian can be obtained from integrations over suitable fermionic coordinates. There are D-term, F-term, and twisted F-term. D-term is for arbitrary superfields  $\mathcal{F}_i$  and is given by

$$\int d^4\theta K(\mathcal{F}, \bar{\mathcal{F}}) = \frac{1}{4} \int d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+ K(\mathcal{F}, \bar{\mathcal{F}}), \quad (2.19)$$

where  $K(\mathcal{F}, \bar{\mathcal{F}})$  is an arbitrary real function of  $\mathcal{F}_i$ 's. This is classically R-invariant under any assignment of R-charges. F-term is for chiral superfields  $\Phi_i$  and is given by

$$\int d^2\theta W(\Phi) + c.c. = \frac{1}{2} \int d\theta^- d\theta^+ W(\Phi)|_{\bar{\theta}^{\pm}=0} + \frac{1}{2} \int d\bar{\theta}^+ d\bar{\theta}^- \bar{W}(\bar{\Phi})|_{\theta^{\pm}=0} \quad (2.20)$$



where  $W(\Phi)$  is a holomorphic function of  $\Phi_i$ 's and is called a superpotential. This is invariant under vector and axial R-symmetries only when it is possible to assign R-charges to  $\Phi_i$ 's so that  $W(\Phi)$  has vector and axial charge 2 and 0 respectively. Twisted F-term is for twisted chiral superfields  $Y_i$  and is given by

$$\int d^2\bar{\theta} \widetilde{W}(Y) + c.c. = \frac{1}{2} \int d\bar{\theta}^- d\theta^+ \widetilde{W}(Y)|_{\bar{\theta}^+ = \theta^- = 0} + \frac{1}{2} \int d\bar{\theta}^+ d\theta^- \widetilde{W}(\bar{Y})|_{\theta^+ = \bar{\theta}^- = 0} \quad (2.21)$$

where  $\widetilde{W}(Y)$  is a holomorphic function of  $Y_i$ 's and is called a twisted superpotential. For R-invariance, it is required that R-charges can be assigned to  $Y_i$ 's so that  $\widetilde{W}(Y_i)$  has vector and axial charge 0 and 2 respectively.

### *Non-linear Sigma Model and Potentials*

Supersymmetric non-linear sigma model on a Kahler manifold  $X$  is one of the main subjects of the present paper. It is described by a set of chiral superfields  $\Phi^i$  ( $i = 1, \dots, n$ ) representing complex coordinates of  $X$ . Let  $g_{i\bar{j}} = \partial^2 K / \partial \phi^i \partial \bar{\phi}^{\bar{j}}$  be the Kahler metric of  $X$  where  $K(\phi, \bar{\phi})$  is a Kahler potential. The Lagrangian can be given by the D-term  $\int d^4\theta K(\Phi, \bar{\Phi})$  which is expressed in terms of the component fields as

$$\begin{aligned} L_K = & -g_{i\bar{j}} \partial^\mu \phi^i \partial_\mu \bar{\phi}^{\bar{j}} + ig_{i\bar{j}} \bar{\psi}_-^{\bar{j}} (D_0 + D_1) \psi_-^i + ig_{i\bar{j}} \bar{\psi}_+^{\bar{j}} (D_0 - D_1) \psi_+^i \\ & + R_{i\bar{k}j\bar{l}} \psi_+^i \psi_-^j \bar{\psi}_-^{\bar{k}} \bar{\psi}_+^{\bar{l}}, \end{aligned} \quad (2.22)$$

after the auxiliary fields are eliminated. Here,  $R_{i\bar{k}j\bar{l}}$  is the curvature tensor with respect to the Levi-Civita connection  $\Gamma_{l\bar{j}}^i = g^{i\bar{k}} \partial_l g_{j\bar{k}}$  of  $X$ . Also,

$$D_\mu \psi_\pm^i = \partial_\mu \psi_\pm^i + \partial_\mu \phi^l \Gamma_{l\bar{j}}^i \psi_\pm^j \quad (2.23)$$

is the covariant derivative with respect to the connection induced on the worldsheet.

If  $X$  has a holomorphic function  $W(\phi)$ , the non-linear sigma model can be deformed by the F-term with superpotential  $W(\Phi)$ ,

$$L = \int d^4\theta K(\Phi, \bar{\Phi}) + \frac{1}{2} \left( \int d^2\theta W(\Phi) + c.c. \right). \quad (2.24)$$

Eliminating the auxiliary fields, we obtain the Lagrangian  $L_K + L_W$  where  $L_K$  is given in (2.22) and the deformation term is

$$L_W = -\frac{1}{4} g^{\bar{j}i} \partial_{\bar{j}} \bar{W} \partial_i W - \frac{1}{2} (D_i \partial_j W) \psi_+^i \psi_-^j - \frac{1}{2} (D_{\bar{i}} \partial_{\bar{j}} \bar{W}) \bar{\psi}_-^{\bar{i}} \bar{\psi}_+^{\bar{j}}, \quad (2.25)$$

in which  $D_i \partial_j W = \partial_i \partial_j W - \Gamma_{ij}^l \partial_l W$ . Note that a non-trivial holomorphic function  $W(\phi)$  exists only when  $X$  is non-compact. In some cases, the deformation discretize the energy

spectrum which would be continuous without  $L_W$  because of the non-compactness. The character of the theory therefore depends largely on the asymptotic behaviour of the superpotential.

If  $X$  has a holomorphic isometry generated by a holomorphic vector field  $V$ , the sigma model can be deformed by another kind of potential term [48]. This is obtained by first gauging the isometry as in [49], taking the weak coupling limit, and freezing the vector multiplet fields at  $\sigma = \widetilde{m}$ ,  $v_\mu = 0$ , and  $\lambda_\pm = \bar{\lambda}_\pm = 0$ . A description in  $(2, 2)$  superspace was considered in [50]. The deformation term is

$$L_V = -g_{i\bar{j}}|\widetilde{m}|^2 V^i \bar{V}^{\bar{j}} - \frac{i}{2} \left( g_{i\bar{i}} \partial_j V^i - g_{j\bar{j}} \partial_{\bar{i}} \bar{V}^{\bar{j}} \right) \left( \widetilde{m} \bar{\psi}_-^i \psi_+^j + \widetilde{m} \bar{\psi}_+^i \psi_-^j \right). \quad (2.26)$$

The deformed Lagrangian is invariant under the modified  $(2, 2)$  supersymmetry where the modification is given by  $\Delta Q_- \bar{\psi}_+^i = -\sqrt{2}i\widetilde{m}\bar{V}^{\bar{i}}$ ,  $\Delta \bar{Q}_+ \psi_-^i = -\sqrt{2}i\widetilde{m}V^i$ , and their complex conjugates (the action on bosonic fields is not modified). The central charge of the modified supersymmetry is non-vanishing and is given by

$$\widetilde{Z} = i\widetilde{m}\mathcal{L}_V \quad (2.27)$$

where  $\mathcal{L}_V$  acts on the fields as  $\mathcal{L}_V \phi^i = V^i$  and  $\mathcal{L}_V \psi_\pm^i = \partial_j V^i \psi_\pm^j$ . If the superpotential  $W(\phi)$  is invariant under the isometry  $V^i \partial_i W = 0$ , then, the sigma model can be deformed by  $L_W$  and  $L_V$  at the same time without breaking  $(2, 2)$  supersymmetry. One can also consider the deformation by a set of commuting holomorphic isometries  $V_1, \dots, V_n$ ; simply replace  $\widetilde{m}V^i \rightarrow \sum_{a=1}^n \widetilde{m}_a V_a^i$ ,  $\widetilde{m}\bar{V}^{\bar{i}} \rightarrow \sum_{a=1}^n \widetilde{m}_a \bar{V}_a^{\bar{i}}$  (the bosonic term  $-|\widetilde{m}|^2|V|^2$  in (2.26) is replaced by  $-\frac{1}{2}|\sum_{a=1}^n \widetilde{m}_a V_a|^2 - \frac{1}{2}|\sum_{a=1}^n \widetilde{m}_a \bar{V}_a|^2$ ).

### *R-Symmetry*

The R-symmetries  $U(1)_V$  and  $U(1)_A$  are not always symmetries of the theory (except their  $\mathbf{Z}_2$  subgroups which acts as the sign flip of spinors —  $2\pi$ -rotation). It can be broken at the classical level by potential terms or at the quantum level by anomaly. However, superconformal field theories always possess both symmetries. We illustrate this in the class of theories introduced above.

We first consider the non-linear sigma model on  $X$  without potentials. Both  $U(1)$ 's are classically unbroken.  $U(1)_V$  remains a symmetry of the quantum theory.  $U(1)_A$  is subject to the chiral anomaly which is proportional to the trace of the curvature of the connection (2.23) of the tangent bundle of  $X$ . Thus,  $U(1)_A$  is anomalous if and only if the first Chern class of  $X$  is non-vanishing;  $c_1(X) \neq 0$ . In particular, both  $U(1)$ 's are unbroken in the sigma model on a CY manifold  $X$ , which is expected to flow to a superconformal field

theory of central charge  $c/3 = \dim X$ . If  $\int \phi^* c_1(X)$  is always an integer multiple of some  $p$ ,  $U(1)_A$  is broken to its discrete subgroup  $\mathbf{Z}_{2p}$  which can be further broken spontaneously. If the theory flows to a non-trivial fixed point,  $U(1)_A$  must be recovered there.

We next consider a theory with a superpotential  $W(\phi)$ .  $U(1)_A$  is classically unbroken but is subject to the chiral anomaly.  $U(1)_V$  is unbroken if and only if  $W(\phi)$  is *scale invariant* in the sense that it is possible to assign the vector R-charges to  $\Phi^i$  so that  $W(\Phi)$  has vector charge 2. The theory has a mass gap if at all the critical points of  $W$  the Hessian is non-degenerate,  $\det \partial_i \partial_j W \neq 0$ . At a degenerate critical point, the theory can flow to a non-trivial fixed point where  $U(1)_V$  is recovered. An example where both  $U(1)$ 's are unbroken is the LG model on  $\mathbf{C}^N$  with a quasi-homogeneous superpotential;  $W(\lambda^{2q_i} \phi_i) = \lambda^2 W(\phi_i)$  where vector R-charge  $2q_i$  is assigned to  $\phi_i$ . It is believed that such a model flows in the IR to an  $N = 2$  superconformal field theory with central charge  $c/3 = \sum_i (1 - 2q_i)$  [7, 8].

Finally, if the sigma model is perturbed by a holomorphic isometry,  $U(1)_A$  is explicitly broken by the fermion mass term in (2.26). This is consistent with the non-vanishing of the susy central charge (2.27). This is also related to the fact that the scalar component of the vector multiplet has a canonical axial charge 2.

### *Supersymmetric Ground States*

Let us examine the ground states of the theory. We compactify the spacial direction on  $S^1$  and put a periodic boundary condition on all fields. We also assume  $Z = \tilde{Z} = 0$ . As in any supersymmetric field theory, a state annihilated by all of  $Q_\pm, \bar{Q}_\pm$  is a zero energy ground state, and vice versa. Let  $Q$  be one of  $Q_- + \bar{Q}_+$  and  $\bar{Q}_+ + \bar{Q}_-$  or their hermitian conjugates. Then, it follows from the algebra (2.1)-(2.4) that

$$\{Q, Q^\dagger\} = 2H. \quad (2.28)$$

It also follows from (2.1)-(2.4) that

$$Q^2 = 0, \quad (2.29)$$

and we can consider the cohomology of states using  $Q$  as the coboundary operator. In the theory where  $H$  has a discrete spectrum, (2.28) and (2.29) imply that the supersymmetric ground states are in one to one correspondence with the  $Q$ -cohomology classes. The index of the operator  $Q$  is the Witten index  $\text{Tr}(-1)^F$  which is invariant under perturbation of the theory.

If the central charge in the supersymmetry algebra is non-vanishing because of a continuous abelian global symmetry group  $T$ , say, as  $\tilde{Z} = \sum_{a=1}^n \lambda_a S_a$  where  $S_a$  are the gener-

ators of  $T$ , (2.28) still holds but (2.29) is modified as  $Q^2 = 2 \sum_{a=1}^n \lambda_a S_a$ . However, when restricted to  $T$ -invariant states,  $Q$  is still nilpotent and we can consider  $Q$ -cohomology. This is the  $T$ -equivariant cohomology. Since a continuous symmetry cannot be broken in  $1+1$  dimensions, the supersymmetric ground states are still in one to one correspondence with the  $T$ -equivariant  $Q$ -cohomology classes.

For a sigma model on a compact Kahler manifold  $X$ ,  $Q = Q_- + \overline{Q}_+$  reduces in the zero momentum sector to the exterior derivative  $d = \partial + \overline{\partial}$  acting on differential forms on  $X$ . In fact, the zero momentum approximation is exact as far as vacuum counting is concerned [51] and the supersymmetric vacua and harmonic forms are in one to one correspondence. In particular  $\text{Tr}(-1)^F = \chi(X)$ . The R-charge of a vacuum corresponding to a harmonic  $(p, q)$ -form is  $q_V = -p + q$  and  $q_A = p + q - \dim_{\mathbf{C}} X$ , where the shift by  $\dim_{\mathbf{C}} X$  is for the invariance of the spectrum under conjugation  $q_A \leftrightarrow -q_A$ . Of course, if  $c_1(X)$  is non-zero and  $U(1)_A$  is anomalous, the axial charge  $q_A$  does not make sense. However, if  $\mathbf{Z}_{2p} \subset U(1)_A$  is non anomalous, there is a  $\mathbf{Z}_{2p}$  grading in the space of vacua.

If  $X$  is non-compact and has a superpotential  $W(\Phi)$ , the spectrum is discrete at sufficiently low energies if  $g^{\bar{j}} \partial_{\bar{j}} \overline{W} \partial_i W \geq c > 0$  at all infinity in the field space. When  $W(\Phi)$  has only non-degenerate critical points, the supersymmetric vacua are in one to one correspondence with the critical points. When  $W(\Phi)$  can be perturbed to such a situation, the index is the number of critical points.

If the compact sigma model is deformed by a holomorphic isometry  $V$ , the susy central charge is non-vanishing and the nilpotency of  $Q = Q_- + \overline{Q}_+$  is modified as  $Q^2 = 2i\widetilde{m}\mathcal{L}_V$ . Since this is a small perturbation, the index remains the same as the  $\widetilde{m} = 0$  case,  $\text{Tr}(-1)^F = \chi(X)$ . In the zero momentum sector,  $Q$  is proportional to  $d_{\widetilde{m}} = d - \sqrt{2i\widetilde{m}} i_V$ . If  $V$  has only non-degenerate zeroes, the supersymmetric vacua are in one to one correspondence with the zeroes of  $V$ . In particular, the index is the number of zeroes (the Hopf index theorem). See [51] for more details.

### *Chiral Ring*

We can also consider cohomology of local operators with respect to  $Q = \overline{Q}_+ + \overline{Q}_-$  or  $Q = Q_- + \overline{Q}_+$  (when the central charge is zero). We shall call a local operator commuting with  $Q_{cc} = \overline{Q}_+ + \overline{Q}_-$  (resp.  $Q_{ac} = \overline{Q}_+ + Q_-$ ) a chiral or *cc* operator (resp. a twisted chiral or *ac* operator). The lowest component of a (twisted) chiral superfield is a (twisted) chiral operator. It follows from the supersymmetry algebra that the space-time translation of a (twisted) chiral operator is  $Q$ -exact and does not change the cohomology class of the operator.

A product of two (twisted) chiral operators is annihilated by  $Q$ . They commute with each other up to  $Q$ -exact operators since one can make them space-like separated. Therefore, the  $Q$ -cohomology group of local operators form a commutative ring [2]. This is called chiral or  $cc$  ring for  $Q_{cc}$ -cohomology and twisted chiral or  $ac$  ring for  $Q_{ac}$ -cohomology. In general  $cc$  and  $ac$  rings in a given theory are different from each other.

If the susy central charge is non-zero because of an abelian global symmetry, we can define equivariant chiral ring in an obvious way.

### *Twisting to Topological Field Theory*

It is often useful to twist  $N = 2$  theories to topological field theories [52]. This is possible when the quantum theory possesses at least either one of  $U(1)_V$  or  $U(1)_A$  R-symmetries (which we call here  $U(1)_R$ , generator  $R$ ) under which the R-charges are all integral. It is standard to call it A-twist for  $R = F_V$  and B-twist for  $R = F_A$ .

We start with the Euclidean version of the theory (obtained by Wick rotation  $x^0 = -ix^2$  from the Minkowski theory). It has the supersymmetry with the same algebra (2.1)-(2.7) and the same hermiticity condition (2.8). Twisting is to replace the group  $U(1)_E$  of space-time rotation generated by  $M$  by the diagonal subgroup of  $U(1)_E \times U(1)_R$ , considering  $M' = M + R$  as the new generator of the rotation group. In particular, the twisted theory on a curved worldsheet is obtained by gauging the diagonal subgroup of  $U(1)_E \times U(1)_R$  (instead of  $U(1)_E$ ) by the spin connection. The energy momentum tensor on the flat worldsheet is thus modified [53–55] as  $T_{\mu\nu}^{\text{twisted}} = T_{\mu\nu} + (1/4)(\epsilon_\mu^\lambda \partial_\lambda J_\nu^R + \epsilon_\nu^\lambda \partial_\lambda J_\mu^R)$  where  $J_\mu^R$  is the  $U(1)_R$  current, and the spin of fields and conserved currents are modified.

An important aspect of the twisted theory is that some of the supercharges have spin zero and make sense without reference to the coordinates. These are  $Q_-$  and  $\overline{Q}_+$  for A-twist while  $\overline{Q}_\pm$  for B-twist (see (2.5)-(2.7)). As we have seen,  $Q_{ac} = Q_- + \overline{Q}_+$  and  $Q_{cc} = \overline{Q}_+ + \overline{Q}_-$  are nilpotent if the susy central charges are zero. Then, we can consider  $Q = Q_{ac}$  or  $Q_{cc}$  as a BRST operator that selects operators in the A-twisted or B-twisted model respectively. In particular,  $ac$  ring elements are the physical operators in the A-model and  $cc$  ring elements are the physical operators in the B-model.  $T_{\mu\nu}^{\text{twisted}}$  is  $Q$ -exact in the class of theories we consider in this paper, and the correlation functions of  $Q$ -invariant operators are independent of the choice of the worldsheet metric. In this sense the twisted theory is a topological field theory. Even if the susy central charge is non-zero because of an abelian global symmetry, the twisted theory can still be considered as topological field theory, physical operators selected by equivariant cohomology.

For the non-linear sigma model possibly with a superpotential, A-twist is possible when  $W(\Phi)$  is scale invariant while B-twist is possible when  $c_1(X) = 0$ . A-twist of a model with  $W \equiv 0$  yields topological sigma model [53] while B-model on a flat manifold  $X$  is called topological LG [56]. The fermion path-integral of a B-model is a chiral determinant and is made well-defined using the anomaly cancellation condition  $c_1(X) = 0$ . The definition involves the choice of a multiplicative factor which can be translated to the choice of a nowhere vanishing holomorphic  $n$ -form where  $n$  is the complex dimension of  $X$ . Indeed,  $c_1(X) = 0$  assures the existence of such an  $n$ -form.

### *Spectral Flow*

Let us return to the untwisted  $N = 2$  theories (where we assume  $Z = \tilde{Z} = 0$  for now). We have seen that supersymmetric ground states of the theory on a periodic circle and the (twisted) chiral ring are both characterized as the cohomology with respect to the nilpotent supercharge  $Q$ . There is in fact an intimate relation between them: *In a theory which can be A-twisted (resp. B-twisted), there is a one-to-one correspondence between supersymmetric ground states and  $ac$  ring elements (resp.  $cc$  ring elements).* In a theory where both A and B-twist are possible, the space of supersymmetric ground states,  $ac$  ring and  $cc$  ring are all the same as a vector space (but of course  $ac$  and  $cc$  rings are different as a ring in general).

This can be seen conveniently using the twisted theory. We consider a theory which is A-twistable. Let us insert an  $ac$  ring element  $\mathcal{O}$  at the tip of a long cigar-like hemisphere in the A-twisted theory. The twisted theory is equivalent to the untwisted  $N = 2$  theory on the flat cylinder region, and we obtain a state of the untwisted theory at the boundary circle. Because of the twisting in the curved region, the fermions are periodic on the boundary circle. Now, the supersymmetric ground state corresponding to  $\mathcal{O}$  is the state at the boundary circle in the limit where the cylinder region becomes infinitely long. The state is indeed a ground state because the infinitely long cylinder plays the role of projection to zero energy states. Various aspects of this relation and the geometry of vacuum states and its relation to the chiral rings have been studied in [57]. In particular this geometry is captured by what is called the  $tt^*$  equations.

## 2.2 The Mirror Symmetry

The  $(2, 2)$  supersymmetry algebra (2.1)-(2.7) is invariant under the outer automorphism given by the exchange of the generators

$$Q_- \leftrightarrow \overline{Q}_-, \quad F_V \leftrightarrow F_A, \quad Z \leftrightarrow \tilde{Z}. \quad (2.30)$$

*Mirror symmetry* is an equivalence of two  $(2, 2)$  supersymmetric field theories under which the generators of supersymmetry algebra are exchanged according to (2.30). A chiral multiplet of one theory is mapped to a twisted chiral multiplet of the mirror, and vice versa. It is of course a matter of convention which to call  $Q_-$  or  $\overline{Q}_-$ . Here, we are assuming the *standard* convention where, in the sigma model on a Kahler manifold (possibly with a superpotential), the complex coordinates are the lowest components of chiral superfields. If we flip the convention of one of a mirror pair, the two theories are equivalent *without* the exchange of (2.30).

### 2.2.1 Mirror Symmetry between Tori

In the case of supersymmetric sigma model on a flat torus, it has been known that mirror symmetry reduces to the  $R \rightarrow 1/R$  duality performed on a middle dimensional torus. Below, we review this in the simplest case of sigma model into the algebraic torus  $\mathbf{C}^\times = \mathbf{R} \times S^1$ . We start with recalling the bosonic  $R \rightarrow 1/R$  duality.

#### $R \rightarrow 1/R$ Duality

Let us consider the following action for a periodic scalar field  $\varphi$  of period  $2\pi$

$$S_\varphi = \frac{1}{4\pi} \int_W R^2 h^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sqrt{h} d^2x, \quad (2.31)$$

where  $h_{\mu\nu}$  is the metric on the worldsheet  $W$  which we choose to be of Euclidean signature. This is the action for a sigma model into a circle  $S^1$  of radius  $R$ . This action can be obtained also from the following action for  $\varphi$  and a one-form field  $B_\mu$

$$S' = \frac{1}{2\pi} \int_W \frac{1}{2R^2} h^{\mu\nu} B_\mu B_\nu \sqrt{h} d^2x + \frac{i}{2\pi} \int_W B \wedge d\varphi. \quad (2.32)$$

Completing the square with respect to  $B_\mu$  which is solved by

$$B = iR^2 * d\varphi, \quad (2.33)$$

and integrating it out, we obtain the action (2.31) for the sigma model.

If, changing the order of integration, we first integrate over the periodic scalar  $\varphi$ , we obtain a constraint  $dB = 0$ . If the worldsheet  $W$  is a genus  $g$  surface, there is a  $2g$ -dimensional space of closed one-forms modulo exact forms<sup>1</sup>. One can choose a basis  $\omega^i$  ( $i = 1, \dots, 2g$ ) such that each element has integral periods on one-cycles on  $W$  and that  $\int_W \omega^i \wedge \omega^j = J^{ij}$  is a non-degenerate matrix of integral entry. Then, a general solution to  $dB = 0$  is

$$B = d\vartheta_0 + \sum_{i=1}^{2g} a_i \omega^i, \quad (2.34)$$

where  $\vartheta_0$  is a real scalar field and  $a_i$ 's are real numbers. Integration over  $\varphi$  actually yields constraints on  $a_j$ 's as well. Recall that  $\varphi$  is a periodic variable of period  $2\pi$ . This means that  $\varphi$  does not have to come back to its original value when circling along a nontrivial one-cycles in  $W$ , but comes back to itself up to  $2\pi$  shifts. For such a topologically nontrivial configuration,  $d\varphi$  has an expansion like (2.34) with non-zero coefficient  $a_i$  for  $\omega^i$  which is dual to the one-cycle. That the shift is only allowed to take integer multiples of  $2\pi$  means that such  $a_i$  is constrained to be  $2\pi n_i$  where  $n_i$  is an integer. Thus, for a general configuration of  $\varphi$  we have

$$d\varphi = d\varphi_0 + \sum_{i=1}^{2g} 2\pi n_i \omega^i, \quad (2.35)$$

where  $\varphi_0$  is a single valued function on  $W$ . Now, integration over  $\varphi$  means integration over the function  $\varphi_0$  and summation over the integers  $n_i$ 's. Integration over  $\varphi_0$  yields the constraint  $dB = 0$  which is solved by (2.34). What about the summation over  $n_i$ 's? To see this we substitute in  $\int B \wedge d\varphi$  for  $B$  from (2.34);

$$\int_W B \wedge d\varphi = 2\pi \sum_{i,j} a_i J^{ij} n_j. \quad (2.36)$$

Now, noting that  $J^{ij}$  is a non-degenerate integral matrix and using the fact that  $\sum_n e^{ian} = 2\pi \sum_m \delta(a - 2\pi m)$ , we see that summation over  $n_i$  constrains  $a_i$ 's to be an integer multiples of  $2\pi$ ;

$$a_i = 2\pi m_i, \quad m_i \in \mathbf{Z}. \quad (2.37)$$

Inserting this into (2.34), we see that  $B$  can be written as

$$B = d\vartheta, \quad (2.38)$$

where now  $\vartheta$  is a periodic variable of period  $2\pi$ . Now, inserting this to the original action we obtain

$$S_\vartheta = \frac{1}{4\pi} \int_W \frac{1}{R^2} h^{\mu\nu} \partial_\mu \vartheta \partial_\nu \vartheta \sqrt{h} d^2x \quad (2.39)$$

---

<sup>1</sup>This can be easily extended to the case of worldsheets with boundaries.



which is an action for a sigma model into  $S^1$  of radius  $1/R$ .

Thus, we have shown that the sigma model into  $S^1$  of radius  $R$  is equivalent to the model with radius  $1/R$ . This is the  $R \rightarrow 1/R$  duality (which is called *target space duality* or *T-duality* in string theory).

Comparing (2.33) and (2.38), we obtain the relation

$$R d\varphi = i \frac{1}{R} * d\vartheta. \quad (2.40)$$

Since  $Rd\varphi$  and  $iR * d\varphi$  are the conserved currents in the original system that count momentum and winding number respectively, the relation (2.40) means that *momentum and winding number are exchanged under the  $R \rightarrow 1/R$  duality*. In particular, the vertex operator

$$\exp(i\vartheta) \quad (2.41)$$

that creates a unit momentum in the dual theory must be equivalent to an operator that creates a unit winding number in the original theory. This can be confirmed by the following path integral manipulation. Let us consider the insertion of

$$\exp\left(-i \int_p^q B\right) \quad (2.42)$$

in the system with the action (2.32), where the integration is along a path  $\tau$  emanating from  $p$  and ending on  $q$ . Then, using (2.38) we see that

$$\exp\left(-i \int_p^q B\right) = e^{-i\vartheta(q)} e^{i\vartheta(p)}. \quad (2.43)$$

On the other hand, the insertion of  $e^{-i \int_p^q B}$  changes the  $B$ -linear term in (2.32). We note that  $\int_p^q B$  can be expressed as  $\int_W B \wedge \omega$ , where  $\omega$  is a one-form with delta function support along the path  $\tau$ . This  $\omega$  can be written as  $\omega = d\theta_\tau$  where  $\theta_\tau$  is a multi-valued function on  $W$  that jumps by one when crossing the path  $\tau$ . Now, the modification of the action (2.32) can be written as

$$\frac{i}{2\pi} \int_W B \wedge d\varphi \longrightarrow \frac{i}{2\pi} \int_W B \wedge d\varphi + i \int_p^q B = \frac{i}{2\pi} \int_W B \wedge d(\varphi + 2\pi\theta_\tau). \quad (2.44)$$

Integrating out  $B_\mu$ , we obtain the action (2.31) with  $\varphi$  replaced by  $\varphi' = \varphi + 2\pi\theta_\tau$ . Note that  $\varphi'$  jumps by  $2\pi$  when crossing the path  $\tau$  which starts and ends on  $p$  and  $q$ . In particular, it has winding number 1 and  $-1$  around  $p$  and  $q$  respectively. Thus, the insertion of  $e^{i\vartheta}$  creates the unit winding number in the original system.

### Mirror Symmetry as $R \rightarrow 1/R$ Duality

We now proceed to a supersymmetric sigma model on the algebraic torus, or the cylinder  $\mathbf{C}^\times = \mathbf{R} \times S^1$ . We show that  $R \rightarrow 1/R$  duality performed on the  $S^1$  factor is indeed a mirror symmetry. We work now in Minkowski signature.

We denote the complex coordinate of the cylinder  $\mathbf{R} \times S^1$  as

$$\phi = \varrho + i\varphi \quad (2.45)$$

where  $\varrho$  is the coordinates of  $\mathbf{R}$  and  $\varphi$  is the periodic coordinate of  $S^1$  of period  $2\pi$ . The Lagrangian of the system is

$$L = \int d^4\theta \frac{R^2}{2} |\Phi|^2 = \frac{R^2}{2} \left( -\eta^{\mu\nu} \partial_\mu \bar{\phi} \partial_\nu \phi + i\bar{\psi}_-(\partial_0 + \partial_1)\psi_- + i\bar{\psi}_+(\partial_0 - \partial_1)\psi_+ \right), \quad (2.46)$$

where  $\Phi$  is the chiral superfield whose lowest component is  $\phi$ . The Kahler metric for  $\phi$  is  $ds^2 = R^2|d\phi|^2 = R^2(d\varrho^2 + d\varphi^2)$  so that  $S^1$  has radius  $R$ .<sup>2</sup>

We perform the duality transformation on  $\varphi$ . As we have seen, this yields another periodic variable  $\vartheta$  of period  $2\pi$  with the Kinetic term  $-(1/2R^2)\eta^{\mu\nu}\partial_\mu\vartheta\partial_\nu\vartheta$ . Thus, the dual theory is also a sigma model into a cylinder, but with a metric

$$d\tilde{s}^2 = d\varrho^2 + \frac{1}{R^2}d\vartheta^2 = \frac{1}{R^2} \left( R^4 d\varrho^2 + d\vartheta^2 \right). \quad (2.47)$$

Thus, either  $R^2\varrho + i\vartheta$  or  $R^2\varrho - i\vartheta$  is the complex coordinates of the new cylinder. What is the superpartner of this (anti-)holomorphic variable? We note the supersymmetry transformations  $\delta\psi_\pm = -i\sqrt{2}(\partial_0 \pm \partial_1)\phi\epsilon^\pm$  and  $\delta\bar{\psi}_\pm = i\sqrt{2}(\partial_0 \pm \partial_1)\phi\epsilon^\pm$ . From (2.40), we see (after continuation back to Minkowski signature by  $x^2 = ix^0$ ) that  $R^2(\partial_0 \pm \partial_1)\varphi = \mp(\partial_0 \pm \partial_1)\vartheta$  and therefore  $R^2(\partial_0 + \partial_1)\phi = (\partial_0 + \partial_1)\eta$  and  $R^2(\partial_0 - \partial_1)\phi = (\partial_0 - \partial_1)\bar{\eta}$  where

$$\eta = R^2\varrho - i\vartheta. \quad (2.48)$$

Thus, the supersymmetry transformation is expressed as

$$R^2\delta\psi_+ = -i\sqrt{2}(\partial_0 + \partial_1)\eta\bar{\epsilon}^+, \quad R^2\delta\bar{\psi}_+ = i\sqrt{2}(\partial_0 + \partial_1)\bar{\eta}\epsilon^+, \quad (2.49)$$

$$R^2\delta\psi_- = -i\sqrt{2}(\partial_0 - \partial_1)\bar{\eta}\epsilon^+, \quad R^2\delta\bar{\psi}_- = i\sqrt{2}(\partial_0 + \partial_1)\eta\epsilon^-. \quad (2.50)$$

This is not a supersymmetry transformation for a chiral multiplet, but that for a twisted chiral multiplet. Indeed, renaming the fermions as

$$R^2\psi_\pm = \pm\bar{\chi}_\pm, \quad R^2\bar{\psi}_\pm = \pm\chi_\pm, \quad (2.51)$$

---

<sup>2</sup>In this paper, we take the convention  $S = \frac{1}{2\pi} \int d^2x L$  as the relation of the action and the Lagrangian. Thus, the weight factor in Path-Integral is  $\exp(\frac{i}{2\pi} \int d^2x L)$  (in Minkowski signature).

the Lagrangian for the dual theory becomes  $\int d^4\theta(-\frac{1}{2R^2}|\Theta|^2)$  for a twisted chiral superfield  $\Theta = \eta + \sqrt{2}(\theta^+\bar{\chi}_+ + \theta^-\chi_-) + \dots$ . Thus, we have seen that  $R \rightarrow 1/R$  duality on  $S^1$  transforms a theory of a chiral multiplet to another theory of a twisted chiral multiplet. Thus, this is a mirror symmetry.

The above manipulation can be simplified by performing the dualization in superspace. We follow the procedure developed in [58]. We start with the following Lagrangian for a real superfield  $B$  and a twisted chiral superfield  $\Theta$ .

$$L' = \int d^4\theta \left( \frac{R^2}{4} B^2 - \frac{1}{2} (\Theta + \bar{\Theta}) B \right) \quad (2.52)$$

We first integrate over the twisted chiral field  $\Theta, \bar{\Theta}$ . This yields the following constraint on  $B$

$$\bar{D}_+ D_- B = D_+ \bar{D}_- B = 0, \quad (2.53)$$

which is solved by

$$B = \Phi + \bar{\Phi}, \quad (2.54)$$

where  $\Phi$  is a chiral superfield. Now, inserting this into the original Lagrangian we obtain the Lagrangian (2.46)

$$L = \int d^4\theta \frac{R^2}{4} (\Phi + \bar{\Phi})^2 = \int d^4\theta \frac{R^2}{2} \bar{\Phi} \Phi. \quad (2.55)$$

for the sigma model into the cylinder with radius  $R$  on  $S^1$ . Now, reversing the order of integration, we consider integrating out  $B$  first. Then,  $B$  is solved by

$$B = \frac{1}{R^2} (\Theta + \bar{\Theta}). \quad (2.56)$$

Inserting this into  $L'$  we obtain

$$\tilde{L} = \int d^4\theta \left( -\frac{1}{2R^2} \bar{\Theta} \Theta \right), \quad (2.57)$$

which is again the Lagrangian for supersymmetric sigma model on the cylinder. This time, the radius of  $S^1$  is  $1/R$  and the complex coordinate is described by the twisted chiral superfield  $\Theta$ . From (2.54) and (2.56), we obtain  $R^2(\Phi + \bar{\Phi}) = \Theta + \bar{\Theta}$  which reproduces the relation between the component fields obtained above (e.g. (2.51)).

### 2.2.2 Examples

Here we present three classes of examples of (conjectural) mirror symmetry. They are mirror symmetry between LG model and LG model, sigma model and sigma model, and sigma model and LG model.

## Minimal Models and Orbifolds

$N = 2$  minimal models are the simplest class of exactly solvable  $(2, 2)$  SCFTs. It has been argued that the  $(d - 2)$ -th minimal model arises as the IR fixed point of the LG model of chiral superfield  $X$  with the superpotential [8, 7]

$$W = X^d. \quad (2.58)$$

If we orbifold the model by the  $\mathbf{Z}_d$  symmetry generated by  $X \rightarrow e^{2\pi i/d} X$ , we obtain again the  $(d - 2)$ -th minimal model [44]

$$\widetilde{W} = \widetilde{X}^d. \quad (2.59)$$

This time, however,  $\widetilde{X}$  is a twisted chiral superfield and  $\widetilde{W}$  is a twisted superpotential. This means that the minimal model and its  $\mathbf{Z}_d$ -orbifold can be considered as mirror to each other.

If a CFT  $\mathcal{C}$  has a discrete abelian symmetry group  $\Gamma$ , the orbifold CFT  $\mathcal{C}' = \mathcal{C}/\Gamma$  has a symmetry group  $\Gamma'$  isomorphic to  $\Gamma$  and the orbifold  $\mathcal{C}'/\Gamma'$  is identical to the original CFT  $\mathcal{C}$ . We apply this general fact to the sum of  $N$  copies of minimal models

$$W = X_1^d + \cdots + X_N^d, \quad (2.60)$$

modulo its  $\mathbf{Z}_d$  symmetry group generated by  $X_i \mapsto X_i \rightarrow e^{2\pi i/d} X_i$  for all  $i$ . This LG orbifold  $\mathcal{C} = (W = \sum_i X_i^d)/\mathbf{Z}_d$  has a symmetry group  $\Gamma \cong (\mathbf{Z}_d)^{N-1}$  generated by  $X_i \mapsto e^{2\pi i\alpha_i/d} X_i$ . Orbifold of  $\mathcal{C}$  by this  $\Gamma$  is  $\mathcal{C}' = (W = \sum_i X_i^d)/(\mathbf{Z}_d)^N = \oplus_i (W = X_i^d)/\mathbf{Z}_d$ . By the mirror symmetry of (2.58) mod  $\mathbf{Z}_d$  and (2.59), this is identical to the sum of  $N$  copies of the minimal model given by the twisted chiral superpotential

$$\widetilde{W} = \widetilde{X}_1^d + \cdots + \widetilde{X}_N^d. \quad (2.61)$$

This indeed has a symmetry  $\Gamma'$  isomorphic to  $(\mathbf{Z}_d)^{N-1}$  generated by  $\widetilde{X}_i \mapsto e^{2\pi i\alpha_i/d} \widetilde{X}_i$  with  $\sum_i \alpha_i = 0 \pmod{d}$ . By the general fact on the orbifold, we see that the orbifold  $(\widetilde{W} = \sum_i \widetilde{X}_i^d)/(\mathbf{Z}_d)^{N-1}$  is identical to the original SCFT  $(W = \sum_i X_i^d)/\mathbf{Z}_d$ . In other words, the  $\mathbf{Z}_d$  orbifold and the  $(\mathbf{Z}_d)^{N-1}$  orbifold of the sum of  $N$  copies of the minimal model are mirror to each other.

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Applying the above construction of mirror pair to the orbifold minimal models [6] corresponding to a CY sigma model (at the special point of its moduli space), Greene and Plesser constructed pairs of CY manifold whose sigma models are mirror to each other [4].

The connection between Landau-Ginzburg models and Calabi-Yau sigma models was first discussed in [9, 8]. The derivation of this connection in [9] involved a change of variables in field space, as if one were dealing with an ordinary integral. This heuristic derivation of the relation between LG models and Calabi-Yau sigma models was made precise by Cecotti [59] who showed the arguments are precise in the context of computing periods associated to special geometry of the LG model, and that the derivation of [9] can be viewed as showing the equivalence of periods and special geometry of the Calabi-Yau with an LG model (in modern terminology as far as the middle dimensional D-brane masses are concerned). In fact this identification of LG models and special geometry associated to the vacuum geometry of the sigma model is crucial in our derivation of mirror symmetry for the case of complete intersections in toric varieties.

The connection between LG models and Calabi-Yau sigma models was further elucidated in [10] using the linear sigma model description, which we will review in section 4 in this paper.

### $\mathbf{CP}^{N-1}$ and Affine Toda Theory

Less known class of mirror symmetry is between non-linear sigma models on manifolds of positive first Chern class and LG models without scale invariance. The typical example is the mirror symmetry of the  $\mathbf{CP}^{N-1}$  sigma model and supersymmetric  $A_{N-1}$  affine Toda theory. The  $\mathbf{CP}^{N-1}$  model is asymptotic free and generates a dynamical scale  $\Lambda$ . It has  $N$  vacua with mass gap.  $U(1)_V$  is unbroken but  $U(1)_A$  is anomalously broken to  $\mathbf{Z}_{2N}$  which is spontaneously broken to  $\mathbf{Z}_2$ . The  $A_{N-1}$  affine Toda theory is an LG model of  $N - 1$  periodic variables  $X_i$  having superpotential

$$W = \Lambda \left( e^{X_1} + \dots + e^{X_{N-1}} + \prod_{i=1}^{N-1} e^{-X_i} \right). \quad (2.62)$$

This theory has the same properties of the  $\mathbf{CP}^{N-1}$  model mentioned above, except that  $U(1)_V$  and  $U(1)_A$  are exchanged, the mass scale is explicitly introduced and  $U(1)_V \rightarrow \mathbf{Z}_{2N}$  is an explicit breaking. This duality is in many ways an  $N = 2$  generalization of the duality of the (bosonic) sine-Gordon theory and the massive Thirring model [60]. In particular, solitons of the affine Toda theory are mapped to the fundamental fields in the  $\mathbf{CP}^{N-1}$  model (in the linear sigma model realization) [61].

The equivalence of the two theories has been observed from various points of view. The agreement of BPS soliton spectrum and their scattering matrix [25, 26], of correlation functions of topologically twisted theories (coupled to topological gravity) [27–29]. This example is extended to a more general class of manifolds in [26, 27, 16, 29].

### 3 The Dynamics of $N = 2$ Gauge Theories in Two Dimensions

In this section, we study the dynamics of  $(2, 2)$  supersymmetric gauge theories in  $1 + 1$  dimensions. We consider  $U(1)$  gauge theory with charged chiral multiplets and assume that there is no superpotential for the charged fields. The theory is described by vector superfield  $V$  with field strength  $\Sigma$  which is a twisted chiral superfield. We consider  $N$  chiral superfields  $\Phi_i$  of charge  $Q_i$ . For earlier studies of this class of theories, see [61, 62, 10].

The classical theory is parametrized by the gauge coupling  $e$ , Fayet-Iliopoulos (FI) parameter  $r$  and Theta angle  $\theta$  where  $e$  has dimension of mass but  $r$  and  $\theta$  are dimensionless. The Lagrangian of the theory is given by

$$L = \int d^4\theta \left( \sum_{i=1}^N \bar{\Phi}_i e^{2Q_i V} \Phi_i - \frac{1}{2e^2} \bar{\Sigma} \Sigma \right) + \frac{1}{2} \left( - \int d^2\tilde{\theta} t \Sigma + c.c. \right), \quad (3.1)$$

where  $t$  is the complex combination

$$t = r - i\theta. \quad (3.2)$$

The theory is super-renormalizable with respect to the gauge coupling  $e$ . However, the FI parameter  $r$  is renormalized to cancel a one-loop divergence unless  $\sum_i Q_i = 0$ . The dependence of the bare parameter  $r_0$  on the cut-off  $\Lambda_{UV}$  is

$$r_0 = \sum_{i=1}^N Q_i \log \left( \frac{\Lambda_{UV}}{\Lambda} \right) \quad (3.3)$$

where  $\Lambda$  is a scale parameter that replaces  $r$  as the parameter of the theory if  $\sum_i Q_i \neq 0$ .

The classical theory is invariant under  $U(1)_V \times U(1)_A$  R-symmetry where  $\Sigma$  is assigned an axial charge 2 and zero vector charge.  $U(1)_V$  is an exact symmetry of the theory but  $U(1)_A$  is subject to the chiral anomaly. The axial rotation by  $e^{i\alpha}$  shifts the theta angle by

$$\theta \rightarrow \theta - 2 \sum_{i=1}^N Q_i \alpha. \quad (3.4)$$

Thus  $U(1)_A$  is unbroken if  $\sum_i Q_i = 0$  but otherwise is broken to the discrete subgroup  $\mathbf{Z}_{2p}$  with  $p = \sum_i Q_i$ .

In addition, the theory has other global symmetries. There are at least  $N - 1$   $U(1)$  symmetries which are the phase rotation of the  $N$  chiral superfield modulo  $U(1)$  gauge transformations. This will be important in our study. Of course there could be larger symmetry if some of the  $U(1)$  charges  $Q_i$  coincide. These global symmetries are non-anomalous and are the symmetries of the quantum theory.

In what follows, we study the dynamics of this gauge theory. We first dualize each of the charged chiral fields  $\Phi_i$  using the phase rotation symmetry. We then study the effective theory described in terms of the dual variables. The goal of this section is to show that the twisted superpotential as mentioned in the introduction of the paper is dynamically generated.

### 3.1 Abelian Duality

Let us consider a complex scalar field  $\phi$  which is minimally coupled to a gauge field  $A_\mu$ . In terms of the polar variables  $(\rho, \varphi)$  defined by  $\phi = \rho e^{i\varphi}$ , the kinetic term  $-\eta^{\mu\nu} D_\mu \phi^\dagger D_\nu \phi$  is written as the sum of  $-(\partial_\mu \rho)^2$  and

$$L_\varphi = -\rho^2 (\partial_\mu \varphi + Q A_\mu)^2, \quad (3.5)$$

where  $Q$  is the charge of  $\phi$ . This Lagrangian is invariant under the shift  $\varphi \rightarrow \varphi + \text{constant}$ , and therefore we can consider dualizing  $\varphi$  as we have done when we discussed  $R \rightarrow 1/R$  duality. What is new here is that we have a gauge field coupled to  $\varphi$  as in (3.5). The dualization procedure start with the following Lagrangian for the vector field  $B_\mu$ , the angle variable  $\varphi$ , plus the gauge field  $A_\mu$ :

$$L' = -\frac{1}{4\rho^2} (B_\mu)^2 + \epsilon^{\mu\nu} B_\mu (\partial_\nu \varphi + Q A_\nu). \quad (3.6)$$

Integration over  $B$  yields the Lagrangian (3.5). If, instead, we first integrate over  $\varphi$ , we obtain the constraint  $B_\mu = \partial_\mu \vartheta$  where  $\vartheta$  is an angle variable of period  $2\pi$ . Plugging this into  $L'$ , we obtain the new Lagrangian

$$L_\vartheta = -\frac{1}{4\rho^2} (\partial_\mu \vartheta)^2 + Q \epsilon^{\mu\nu} \partial_\mu \vartheta A_\nu = -\frac{1}{4\rho^2} (\partial_\mu \vartheta)^2 - Q \vartheta \epsilon^{\mu\nu} \frac{1}{2} F_{\mu\nu}, \quad (3.7)$$

where a partial integration is used. Thus, the dual variable  $\vartheta$  is coupled to the gauge field  $A_\mu$  as a dynamical Theta angle.

#### *Supersymmetric Case*

It is straightforward to repeat this dualization in the supersymmetric theory with a chiral superfield  $\Phi$  of charge  $Q$ . In fact, we only have to dualize the phase of  $\Phi$  and suitably rename other fields. As we have seen above, the dual variable couples to the gauge field as a dynamical Theta angle. Such a variable must be in a twisted chiral multiplet  $Y$  that couples to the field strength  $\Sigma$  in the twisted superpotential as

$$Q Y \Sigma. \quad (3.8)$$

One can see this explicitly by performing the duality transformation in the superspace, as we now show.

We start with the following Lagrangian for a vector superfield  $V$ , a real superfield  $B$  and a twisted chiral superfield  $Y$  whose imaginary part is periodic with period  $2\pi$ .

$$L' = \int d^4\theta \left( e^{2QV+B} - \frac{1}{2}(Y + \bar{Y})B \right), \quad (3.9)$$

where  $Q$  is an integer. We first integrate over  $Y$ . This yields the constraint  $\bar{D}_+ D_- B = D_+ \bar{D}_- B = 0$  on  $B$  which is solved by

$$B = \Psi + \bar{\Psi}, \quad (3.10)$$

where  $\Psi$  is a chiral superfield. Since the imaginary part of  $Y$  is an angular variable of period  $2\pi$ , so is the imaginary part of  $\Psi$ . Now, inserting this into the original Lagrangian we obtain

$$L = \int d^4\theta e^{2QV+\Psi+\bar{\Psi}} \quad (3.11)$$

which is nothing but the Lagrangian for the chiral superfield  $\Phi = e^\Psi$  of charge  $Q$ . Now, reversing the order of integration, we consider integrating out  $B$  first. Then,  $B$  is solved by

$$B = -2QV + \log \left( \frac{Y + \bar{Y}}{2} \right). \quad (3.12)$$

Inserting this into  $L'$  we obtain

$$\tilde{L} = \int d^4\theta \left( QV(Y + \bar{Y}) - \frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) \right) \quad (3.13)$$

Using the fact that  $Y$  is a twisted chiral superfield,  $\bar{D}_+ Y = D_- Y = 0$ , the term proportional to  $V$  can be written as

$$\int d^4\theta VY = -\frac{1}{4} \int d\theta^+ d\bar{\theta}^- \bar{D}_+ D_- V Y \quad (3.14)$$

$$= \frac{1}{2} \int d^2\tilde{\theta} \Sigma Y \quad (3.15)$$

where we have used  $\Sigma = \bar{D}_+ D_- V$  which holds for abelian gauge group. Together with the gauge kinetic term and the classical FI-Theta terms, we obtain the following Lagrangian<sup>1</sup>:

$$\tilde{L} = \int d^4\theta \left\{ -\frac{1}{2e^2} \bar{\Sigma} \Sigma - \frac{1}{2}(Y + \bar{Y}) \log(Y + \bar{Y}) \right\} + \frac{1}{2} \left( \int d^2\tilde{\theta} \Sigma (QY - t) + c.c. \right) \quad (3.16)$$

---

<sup>1</sup>In the usual T-duality the dilaton shifts, proportional to the volume of the space. In the case at hand, since translations in the dualizing circle is gauged this shift in dilaton does not arise.



We indeed see that the charged chiral superfield  $\Phi$  has turned into a neutral twisted chiral superfield  $Y$  which couples to the field strength  $\Sigma$  as the dynamical Theta angle.

It follows from (3.10) and (3.12) that the original chiral field  $\Phi$  and the twisted chiral field  $Y$  are related by

$$Y + \bar{Y} = 2\bar{\Phi} e^{2QV} \Phi. \quad (3.17)$$

We see that the dual field  $Y$  is a gauge invariant composite of the original field  $\Phi$ . Using the expression (2.17) of  $V$  in the Wess-Zumino gauge, we can write down the relation between the components fields  $y = \varrho - i\vartheta, \bar{\chi}_+, \chi_-$  of  $Y$  and those  $\phi = \rho e^{i\varphi}, \psi_+, \psi_-$  of  $\Phi$ :

$$\varrho = \rho^2, \quad (3.18)$$

$$\partial_{\pm}\vartheta = \pm 2 \left( -\rho^2(\partial_{\pm}\varphi + QA_{\pm}) + \bar{\psi}_{\pm}\psi_{\pm} \right) \quad (3.19)$$

where  $\partial_{\pm} = \partial_0 \pm \partial_1$  etc, and

$$\bar{\chi}_+ = 2\phi^{\dagger}\psi_+, \quad \chi_- = -2\bar{\psi}_-\phi, \quad (3.20)$$

$$\chi_+ = 2\bar{\psi}_+\phi, \quad \bar{\chi}_- = -2\phi^{\dagger}\psi_-. \quad (3.21)$$

We note that the term  $\pm 2\bar{\psi}_{\pm}\psi_{\pm}$  in (3.19) reflects the fact that we are dualizing on the phase of the whole superfield  $\Phi$ . The Kahler metric of the field  $Y$  is given by

$$ds^2 = \frac{|dy|^2}{2(y + \bar{y})} = \frac{1}{4\varrho} (d\varrho^2 + d\vartheta^2) = d\rho^2 + \frac{1}{4\rho^2} d\vartheta^2, \quad (3.22)$$

as can also be seen from the bosonic treatment (e.g. (3.7)). We note that the relation (3.18) implies a condition that the real part of  $y$  is allowed to take only non-negative values,  $\text{Re}(y) \geq 0$ . The boundary  $\text{Re}(y) = 0$  corresponds to  $|\Phi| = 0$  where the circle on which we are dualizing shrinks to zero size. With respect to the metric (3.22), the boundary is at finite distance from any point with finite  $\text{Re}(y)$ . Naively we expect a lot of singularities coming from such end points. However, it will be argued below that the physically relevant region is infinitely far away from such a boundary region, once the renormalization is taken into account.

### *Renormalization*

We recall that we had to renormalize the FI parameter of the theory as (3.3). In order for the coupling  $\Sigma(QY - t)$  to be finite, we have to renormalize also the field  $Y$ . This can be done by letting the bare dual field  $Y_0$  to depend on the UV cut-off as

$$Y_0 = \log(\Lambda_{UV}/\mu) + Y, \quad (3.23)$$

where  $\mu$  is the scale at which the field is renormalized.

Note that this resolves the issue of the bound  $\text{Re}(y) \geq 0$ . In fact, the correct condition is  $\text{Re}(y_0) \geq 0$  and therefore it means

$$\text{Re}(y) \geq -\log(\Lambda_{UV}/\mu) \quad (3.24)$$

for the renormalized variable. In particular, in the continuum limit  $\Lambda_{UV}/\mu \rightarrow \infty$ , there is no bound on the renormalized field. Also, we note that the Kahler metric for the renormalized field is

$$ds^2 = \frac{|dy|^2}{2(2\log(\Lambda_{UV}/\mu) + y + \bar{y})} \simeq \frac{|dy|^2}{4\log(\Lambda_{UV}/\mu)} \quad (3.25)$$

which becomes flat in the continuum limit.

### *R-Symmetry*

We would like to know the transformation property of the new field  $Y$  under the vector and axial R-symmetries. It appears from the relation (3.17) that the superfield  $Y$  transforms as a charge zero field under both  $U(1)_V$  and  $U(1)_A$ . However, it cannot be directly seen from (3.17) how the imaginary part  $\vartheta$  of  $Y$  transforms. Nevertheless, one can read the transformation of  $\vartheta$  from (3.17) or (3.19) in a way similar to [34]. Let us note that the conserved currents of the vector and axial R-symmetries are given by

$$J_{\pm}^V = \bar{\psi}_{\pm}\psi_{\pm} + \cdots, \quad J_{\pm}^A = \pm\bar{\psi}_{\pm}\psi_{\pm} + \cdots, \quad (3.26)$$

where  $+\cdots$  are contributions from the vector multiplet fields. The operator product of these with  $\partial_{\pm}\vartheta$  expressed as (3.19) has the following singularity:

$$J_{\pm}^V(x)\partial_{\pm}\vartheta(y) \sim \frac{\pm 2}{(x^{\pm} - y^{\pm})^2}, \quad (3.27)$$

$$J_{\pm}^A(x)\partial_{\pm}\vartheta(y) \sim \frac{2}{(x^{\pm} - y^{\pm})^2}. \quad (3.28)$$

These show that  $\vartheta$  is invariant under vector R-symmetry but is shifted by 2 by the axial R-symmetry. Therefore the superfield  $Y$  transforms under  $U(1)_V \times U(1)_A$  as

$$e^{i\alpha F_V} Y(\theta^{\pm}, \bar{\theta}^{\pm}) e^{-i\alpha F_V} = Y_i(e^{-i\alpha}\theta^{\pm}, e^{i\alpha}\bar{\theta}^{\pm}), \quad (3.29)$$

$$e^{i\alpha F_A} Y(\theta^{\pm}, \bar{\theta}^{\pm}) e^{-i\alpha F_A} = Y(e^{\mp i\alpha}\theta^{\pm}, e^{\pm i\alpha}\bar{\theta}^{\pm}) - 2i\alpha. \quad (3.30)$$

Indeed, the Lagrangian (3.16) exhibits the  $U(1)_V$  invariance and the correct  $U(1)_A$  anomaly under this action.

### Multi-Flavor Case

It is straightforward to extend the dualization considered above to the case where there are several charged chiral fields. Dualizing each of the chiral fields  $\Phi_i$ , we obtain the following twisted superpotential for the twisted chiral fields  $Y_{i0}$

$$\widetilde{W} = \Sigma \left( \sum_{i=1}^N Q_i Y_{i0} - t_0 \right). \quad (3.31)$$

The relation between  $\Phi_i$  and  $Y_{i0}$  are the same as in (3.17)-(3.21). We renormalize the fields as

$$Y_{i0} = \log(\Lambda_{UV}/\mu) + Y_i. \quad (3.32)$$

so that (3.31) is finite in the continuum limit  $\Lambda_{UV} \rightarrow \infty$  in the case  $\sum_i Q_i \neq 0$ . In the case  $\sum_i Q_i = 0$ , (3.31) is invariant under an  $i$ -independent shift of  $Y_{i0}$ 's and we also do this field redefinition (3.32). In any case, the bound  $\text{Re}(y_{i0}) \geq 0$  is eliminated from  $y_i$ 's. With respect to the renormalized fields the twisted superpotential can be written as

$$\widetilde{W} = \Sigma \left( \sum_{i=1}^N Q_i Y_i - t(\mu) \right). \quad (3.33)$$

where  $t(\mu)$  is the effective FI-Theta parameter

$$t(\mu) = \begin{cases} \sum_{i=1}^N Q_i \log(\mu/\Lambda) - i\theta & \text{if } \sum_{i=1}^N Q_i \neq 0, \\ r - i\theta & \text{if } \sum_{i=1}^N Q_i = 0. \end{cases} \quad (3.34)$$

As in the single flavor case, one can see that the R-symmetry group  $U(1)_V \times U(1)_A$  acts on the fields  $Y_i$  as

$$e^{i\alpha F_V} Y_i(\theta^\pm, \bar{\theta}^\pm) e^{-i\alpha F_V} = Y_i(e^{-i\alpha} \theta^\pm, e^{i\alpha} \bar{\theta}^\pm), \quad (3.35)$$

$$e^{i\alpha F_A} Y_i(\theta^\pm, \bar{\theta}^\pm) e^{-i\alpha F_A} = Y_i(e^{\mp i\alpha} \theta^\pm, e^{\pm i\alpha} \bar{\theta}^\pm) - 2i\alpha. \quad (3.36)$$

We see that the superpotential (3.33) is invariant under  $U(1)_V$ . Note also that (3.33) exhibits the axial anomaly (3.4) for  $p = \sum_i Q_i \neq 0$  and the breaking of  $U(1)_A$  down to  $\mathbf{Z}_{2p}$ . It may appear that there are extra symmetries  $Y_i \rightarrow Y_i + c_i$  with  $c_i \neq c_j$  and  $\sum_{i=1}^N Q_i c_i = 0$  that makes the transformation (3.36) ambiguous. However, there is no corresponding symmetry in the original system;  $Y_i$ 's can be shifted only by axial symmetries and the only axial symmetry in the system is the  $U(1)_A$  with current  $J_\pm^A = \pm \sum_{i=1}^N \bar{\psi}_{i\pm} \psi_{i\pm} + \dots$  as long as  $Q_i$ 's are all non-zero. Thus, there is no room for ambiguity in the R-transformation (3.36). In fact, new terms in  $\widetilde{W}$  that violates the invariance under the extra shifts with  $c_i \neq c_j$  is generated, as we will show next.

### 3.2 Dynamical Generation Of Superpotential

The twisted superpotential (3.33) obtained from dualization is an exact expression in perturbation theory with respect to  $1/r$ ; it is simply impossible to write down a perturbative correction that respects the R-symmetry and/or anomaly, holomorphy in  $t$ , and periodicity of Theta angle. The D-term is of course subject to perturbative corrections.

However, the twisted superpotential is possibly corrected by non-perturbative effects. A typical non-perturbative effect in quantum field theory is by the presence of instantons. The bosonic part of our theory is an abelian Higgs model which can have an instanton configuration — vortex. It has been known that in an abelian Higgs model a Theta dependent vacuum energy density is generated by the effect of the gas of vortices and anti-vortices [63]. As in that case, and also as in Polyakov's model of confinement where a bosonic potential for the dual field is generated from the gas of monopoles and anti-monopoles, we expect that a superpotential for  $Y_i$ 's can be generated by the gas of vortices and anti-vortices.

Around the vortex for a charged scalar  $\phi_i$ , the phase of  $\phi_i$  has winding number one. As we have seen in  $R \rightarrow 1/R$  duality, a winding configuration is dual to the insertion of the vertex operator  $e^{i\vartheta_i}$ . The supersymmetric completion of this operator is the twisted chiral superfield

$$e^{-Y_i}. \quad (3.37)$$

These exponentials have vector R-charge 0 and axial R-charge 2, as can be seen from (3.35) and (3.36). Thus, we can add these to the twisted superpotential without violating the  $U(1)_V$  R-symmetry, and maintaining the correct anomaly of  $U(1)_A$ .

In what follows we shall show that a correction of the form (3.37) is indeed generated. In fact we will show that the correction is simply the sum of them and the exact superpotential is given by

$$\widetilde{W} = \Sigma \left( \sum_{i=1}^N Q_i Y_i - t(\mu) \right) + \mu \sum_{i=1}^N e^{-Y_i}. \quad (3.38)$$

This is one of the main results of this paper. Note that the change of the renormalization scale  $\mu$  can be absorbed by the shift of  $Y_i$ 's dictated by (3.32). The actual parameter of the theory is still the dynamical scale  $\Lambda$  for  $\sum_i Q_i \neq 0$  and the FI-Theta parameter  $t$  for  $\sum_i Q_i = 0$ .

Before embarking on the computation to show that  $e^{-Y_i}$ 's are indeed generated, we make a simple consistency check of (3.38). Let us consider integrating out  $Y_i$ 's for a fixed configuration of  $\Sigma$  whose lowest component of  $\Sigma$  is large and slowly varying. The variation

with respect to  $Y_i$ 's yields the relation  $Q_i \Sigma - \mu e^{-Y_i} = 0$  or  $Y_i = -\log(Q_i \Sigma / \mu)$ . Inserting this into (3.38) we obtain the effective superpotential for the  $\Sigma$  field

$$\widetilde{W}_{\text{eff}}(\Sigma) = \Sigma \left( - \sum_{i=1}^N Q_i \left( \log(Q_i \Sigma / \mu) - 1 \right) - t(\mu) \right). \quad (3.39)$$

This is nothing but what we would obtain when we integrate out the chiral superfield  $\Phi_i$ 's in the original gauge theory for a fixed configuration of  $\Sigma$  [64, 26, 10, 65].

### 3.2.1 Localization

We first establish that the validity of (3.38) for general  $N$  and  $Q_i$  is a consequence of the case with  $N = 1$ . This can be seen by considering a theory with a larger gauge symmetry where the  $U(1)^{N-1}$  global symmetries are gauged and recovering the original theory in the weak coupling limit of the extra gauge interactions. Sometimes we will refer to this procedure as *localization*.

We start with the  $U(1)^N$  gauge theory with a single charged matter for each  $U(1)$  which is described by the following classical Lagrangian

$$L = - \sum_{i,j} \int d^4\theta \frac{1}{2e_{ij}^2} \bar{\Sigma}_i \Sigma_j + \sum_{i=1}^N \left\{ \int d^4\theta \bar{\Phi}_i e^{2Q_i V_i} \Phi_i + \frac{1}{2} \left( - \int d^2\tilde{\theta} t_i \Sigma_i + c.c. \right) \right\}. \quad (3.40)$$

If the gauge coupling matrix is diagonal

$$\frac{1}{e_{ij}^2} = \delta_{i,j} \frac{1}{e_i^2}, \quad (3.41)$$

the  $N$  single flavor theories are decoupled from each other. In such a case, the exact twisted superpotential is given by the sum of those for single flavor theories. If we assume that (3.38) holds for the single flavor cases, it is

$$\widetilde{W} = \sum_{i=1}^N \left( \Sigma_i \left( Q_i Y_i - t_i(\mu) \right) + \mu e^{-Y_i} \right), \quad (3.42)$$

where we have chosen the scale  $\mu$  common to all  $i$ , and  $t_i(\mu)$  is the effective FI-Theta parameter at  $\mu$

$$t_i(\mu) = Q_i \log(\Lambda_i / \mu) - i \theta_i, \quad (3.43)$$

for the  $i$ -th theory.

Now the essential point here is that the (twisted) superpotential is independent of variation of the D-term. Thus, (3.42) is valid for all values of  $1/e_{ij}^2$  as long as there is

no singularity in going from the diagonal one (3.41). In particular, let us consider the coupling matrix of the following form

$$\sum_{i,j} \frac{1}{2e_{ij}^2} \bar{\Sigma}_i \Sigma_j = \frac{1}{2e^2} \left| \frac{1}{N} \sum_{i=1}^N \Sigma_i \right|^2 + \frac{1}{\epsilon e^2} \sum_{i=1}^{N-1} |\Sigma_i - \Sigma_{i+1}|^2. \quad (3.44)$$

In the limit

$$\epsilon \rightarrow 0, \quad (3.45)$$

the only dynamical gauge symmetry is the diagonal  $U(1)$  subgroup of  $U(1)^N$  with the field strength given by

$$\Sigma_1 + \Sigma_2 + \cdots + \Sigma_N =: N\Sigma, \quad (3.46)$$

and we recover the  $U(1)$  gauge theory with  $N$  matter fields  $\Phi_i$  with charge  $Q_i$ . To be precise, the limit (3.45) itself does not completely fix  $\Sigma_i - \Sigma_j$ , but rather sets it to be a constant. In other words we have the choice  $\Sigma_i = \Sigma + \Delta_i$  with  $\Delta_i$  being a constant. However, such a shift  $\Delta_i$  would yield a twisted superpotential whose perturbative part does not match with what we have (3.33) for the single  $U(1)$  gauge theory. Therefore we must have  $\Delta_i = 0$  in the present theory. Nevertheless as we will discuss later a non-zero  $\Delta_i$  is indeed allowed when we consider a perturbation of the theory with “twisted masses”. Therefore we will in general take into account the above possible deformation of the  $U(1)$  gauge theory.

The bare FI and Theta parameter of the  $U(1)$  gauge theory is related to those of the  $U(1)^N$  theories by

$$t_0 = \sum_{i=1}^N (Q_i \log(\Lambda_{UV}/\Lambda_i) - i\theta_i). \quad (3.47)$$

In particular, for the theory with  $\sum_{i=1}^N Q_i \neq 0$ , the dynamical scale  $\Lambda$  is given by  $\prod_i \Lambda^{Q_i} = \prod_i \Lambda_i^{Q_i}$ , while the FI parameter of the theory with  $\sum_{i=1}^N Q_i = 0$  is  $r = -\sum_{i=1}^N Q_i \log(\Lambda_i)$ . Then, the effective coupling  $t(\mu)$  at energy  $\mu$  is simply the sum  $\sum_{i=1}^N t_i(\mu)$ . Thus, the twisted superpotential (3.42) becomes (3.38) in the limit (3.45). This shows that (3.38) for general  $N$  and  $Q_i$  follows from the  $N = 1$  case.

Thus, to show (3.38) we only have to show the single flavor case. We note also that in the single flavor case we only have to show it in the case of unit charge  $Q = 1$ . Other cases just follow from that case by a redefinition of the gauge field and the FI-Theta parameter;  $Q\Sigma \rightarrow \Sigma$ ,  $Qt \rightarrow t$ .

### 3.2.2 The Generation Of Superpotential

Now, let us consider the single flavor case with  $Q = 1$ . We first determine the possible form of the non-perturbative correction  $\Delta\widetilde{W}$  from the general requirements [66] — holomorphy in  $t$ , periodicity in Theta angle, R-symmetry, and asymptotic behaviour. Let  $\Lambda = \mu e^{-t}$  be the dynamical scale and let us put  $\widetilde{Y} = Y - t$ . Since  $t$  and  $Y$  are periodic with period  $2\pi i$ ,  $\Delta\widetilde{W}$  must be a holomorphic function of  $\Sigma$ ,  $\Lambda$  and  $e^{-\widetilde{Y}}$ . The anomaly (3.4) of the axial R-symmetry is absorbed by the shift of the Theta angle  $\theta \rightarrow \theta + 2\alpha$ , or  $t \rightarrow t - 2i\alpha$ . This modified  $U(1)_A$  symmetry transforms the variables as

$$\Sigma \rightarrow e^{2i\alpha}\Sigma, \quad \Lambda \rightarrow e^{2i\alpha}\Lambda, \quad e^{-\widetilde{Y}} \rightarrow e^{-\widetilde{Y}}. \quad (3.48)$$

Since the twisted superpotential must have charge 2 under this transformation,  $\Delta\widetilde{W}$  must be of the form  $\Sigma f(\Lambda/\Sigma, e^{-\widetilde{Y}})$  which is expanded in a Laurent series as

$$\Delta\widetilde{W} = \Sigma \sum_{n,m} c_{n,m} (\Lambda/\Sigma)^n e^{-m\widetilde{Y}} = \sum_{n,m} c_{n,m} \Sigma^{1-n} \mu^n e^{-(n-m)t - mY}. \quad (3.49)$$

Now, we recall that the field  $Y$  was introduced by dualizing the circle of radius  $|\phi|$ . Therefore, in the description in terms of  $Y$  and  $\Sigma$ , we are looking at the region  $\phi \neq 0$  of the field space where the gauge symmetry is broken and  $\Sigma$  is therefore massive. Thus, the twisted superpotential must be analytic at  $\Sigma = 0$ . This means that only terms of  $1 - n \geq 0$  is non-vanishing in (3.49). On the other hand, in the semi-classical limit where  $r$  is very large, the correction must be small compared to the perturbative term  $\Sigma(Y - t)$ . This requires  $n \geq m$ . Finally, since  $Y$  is unbounded in real positive direction (but is bounded from below as  $\text{Re}Y \sim |\Phi|^2 \geq 0$  in the semi-classical description),  $m \geq 0$  is also required for the correction to be small. To summarize, only  $1 \geq n \geq m \geq 0$  is allowed.  $n = m = 0$  is of the same order as the perturbative term and is excluded.  $n = 1, m = 0$  is just a constant term. The final candidate  $n = m = 1$  is a non-trivial term which is  $e^{-Y}$ . Thus, the twisted superpotential must be of the form

$$\widetilde{W} = \Sigma(Y - t(\mu)) + c\mu e^{-Y}, \quad (3.50)$$

where  $c$  is a dimensionless constant.

The question is therefore whether the coefficient  $c$  is zero or not. We have already observed an evidence that supports non-vanishing of  $c$ ; Integration over  $Y$  yields the correct effective superpotential for  $\Sigma$ ,  $\widetilde{W}_{\text{eff}}(\Sigma) = \Sigma \log(\Sigma/\Lambda)$ , if  $c$  is non-zero. In what follows we show that the term  $\mu e^{-Y}$  is indeed generated by an instanton effect.

## The Vortex

We continue our gauge theory to the Euclidean signature by Wick rotation  $x^0 = -ix^2$ . We choose the orientation so that  $z = x^1 + ix^2$  is the complex coordinates. This leads to  $F_{01} \rightarrow -iF_{12}$ ,  $D_0 + D_1 \rightarrow 2D_{\bar{z}}$  and  $D_0 - D_1 \rightarrow -2D_z$ . After solving for the auxiliary field as  $D = -e^2(|\phi|^2 - r_0)$  and  $F = 0$ , the Euclidean action is given by

$$\begin{aligned}
S_E = \frac{1}{2\pi} \int d^2x \left( |D_\mu \phi|^2 + |\sigma \phi|^2 + \frac{1}{2e^2} |\partial_\mu \sigma|^2 + \frac{1}{2e^2} (F_{12}^2 + D^2) + i\theta F_{12} \right. \\
- 2i\bar{\psi}_- D_{\bar{z}} \psi_- + 2i\bar{\psi}_+ D_z \psi_+ + \bar{\psi}_- \sigma \psi_+ + \bar{\psi}_+ \bar{\sigma} \psi_- \\
+ \frac{1}{e^2} \left( -i\bar{\lambda}_- \partial_{\bar{z}} \lambda_- + i\bar{\lambda}_+ \partial_z \lambda_+ \right) \\
\left. + i \left( \phi^\dagger \lambda_- \psi_+ - \phi^\dagger \lambda_+ \psi_- - \bar{\psi}_+ \bar{\lambda}_- \phi + \bar{\psi}_- \bar{\lambda}_+ \phi \right) \right). \quad (3.51)
\end{aligned}$$

An instanton is a topologically non-trivial configuration that minimizes the bosonic part of this action.

An instanton can contribute to the (twisted) superpotential only when it carries two fermionic zero modes of the right kind. Since a twisted F-term is obtained by the integration over two fermionic coordinates other than  $\theta^-$  and  $\bar{\theta}^+$ , a relevant configuration must be invariant under the supercharges  $Q_-$  and  $\bar{Q}_+$ . The invariance of the fermions under these supercharges requires (see Appendix B)

$$\sigma = 0, \quad (3.52)$$

$$D_{\bar{z}} \phi = 0, \quad (3.53)$$

$$F_{12} = e^2(|\phi|^2 - r_0). \quad (3.54)$$

The bosonic part of the action is  $\frac{1}{2\pi} \int d^2x (\frac{1}{2e^2} |\partial_\mu \sigma|^2 + |\sigma \phi|^2)$  plus

$$\begin{aligned}
& \frac{1}{2\pi} \int d^2x \left( |D_\mu \phi|^2 + \frac{1}{2e^2} (F_{12}^2 + D^2) + i\theta F_{12} \right) \\
&= \frac{1}{2\pi} \int d^2x \left( |2D_{\bar{z}} \phi|^2 - F_{12} |\phi|^2 + \frac{1}{2e^2} (F_{12} + D)^2 - \frac{1}{e^2} D F_{12} + i\theta F_{12} \right) \\
&= \frac{1}{2\pi} \int d^2x \left( |2D_{\bar{z}} \phi|^2 + \frac{1}{2e^2} (F_{12} + D)^2 \right) - \frac{t_0}{2\pi} \int F_{12} d^2x, \quad (3.55)
\end{aligned}$$

where  $D = -e^2(|\phi|^2 - r_0)$  and  $t_0 = r_0 - i\theta$ . For a given topological number

$$k = -\frac{1}{2\pi} \int F_{12} d^2x, \quad (3.56)$$

the real part of the action is bounded by  $kr_0$ , and the minimum is indeed attained by a solution to the equations (3.52)-(3.54). The value of the action for such an instanton is

$$S_E = kt_0. \quad (3.57)$$



Under the axial rotation by  $e^{i\alpha}$ , the path-integral measure in this topological sector changes by the phase  $e^{2ik\alpha}$ . Since the twisted superpotential has axial R-charge 2, we see that the relevant configurations are those with  $k = 1$ .

A solution to (3.53) and (3.54) is the vortex. For each vortex with  $k = 1$ ,  $\phi$  has a single simple zero. The moduli space of gauge equivalence classes of  $k = 1$  vortices is complex one-dimensional and is parametrized by the location of the zero of  $\phi$ . To see this, we note the following well-known fact. The orbit of a solution to (3.53) under the complexified gauge transformations contains vortex solutions in one gauge equivalence class, and conversely, any gauge equivalence class of vortex solutions is contained in one such orbit. Here, a complexified gauge transformation is a rotation  $(iA_{\bar{z}}, \phi) \rightarrow (iA_{\bar{z}} + h\partial_{\bar{z}}h^{-1}, h\phi)$  by a function  $h$  with values in  $\mathbf{C}^\times$ . Thus, we only have to find solutions to the equation  $D_{\bar{z}}\phi = 0$  modulo the complexified gauge transformations. In other words, we only have to find pairs of a holomorphic line bundle with a holomorphic section. Here, it is convenient to compactify our Euclidean 2-plane to a Riemann sphere. Then, there is a unique holomorphic line bundle of  $k = 1$ . Such a bundle has two-dimensional space of holomorphic sections, where each section has a single simple zero. The residual complexified gauge symmetry is a multiplication by a constant and it does not change the location of the zero of a section. Thus, the space of equivalence classes is complex one-dimensional and is parametrized by the zero locus of  $\phi$ .

Let us examine in more detail the behaviour of a vortex solution. We consider the vortex at  $z = 0$ . For the finiteness of the action,  $|\phi|$  must approach the vacuum value  $|\phi|^2 = r_0$  at infinity. By rescaling  $\phi = \sqrt{r_0}\hat{\phi}$  where  $|\hat{\phi}| \rightarrow 1$  at infinity, it becomes clear that the equations depend only on one length parameter  $1/e\sqrt{r_0}$ . This characterizes the size of the vortex. Thus, we expect that the gauge field is nearly flat on  $|z| \gg 1/e\sqrt{r_0}$  and  $\hat{\phi}$  is nearly covariantly constant there. Since  $(-1/2\pi) \int F_{12} = 1$ , we have  $A_\mu = -\partial_\mu \arg(z)$  and  $\phi = \sqrt{r_0}z/|z|$  at infinity. The exact solution takes the form

$$iA = -\frac{1-f}{2} \left( \frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right), \quad (3.58)$$

$$\phi = \sqrt{r_0} \exp \left( - \int_{|z|^2}^{\infty} \frac{dw f(w)}{2w} \right) \frac{z}{|z|}, \quad (3.59)$$

where  $f$  is a function of  $w = |z|^2$  which satisfies the equation  $wf'' = \frac{e^2 r_0}{2} f + f f'$  and the boundary condition  $f(0) = 1, f(+\infty) = 0$ . The asymptotic behaviour of the function  $f(|z|^2)$  at  $|z| \gg 1/e\sqrt{r_0}$  is

$$f = \text{const} \sqrt{m|z|} e^{-m|z|} + \dots, \quad (3.60)$$

where *const* is a numerical constant and

$$m = e\sqrt{2r_0}. \quad (3.61)$$

The vortex at  $z = z_0$  is obtained from the above solution simply by the replacement  $z \rightarrow z - z_0$ .

### *Fermionic Zero Modes*

Now let us examine the fermionic zero modes in the instanton background. Since  $\sigma = 0$  in this background, the fermionic part of the action (3.51) decomposes into two parts as

$$-i(\bar{\psi}_-, \lambda_+) \begin{pmatrix} 2D_{\bar{z}} & -\phi \\ \phi^\dagger & -\frac{1}{e^2}\partial_z \end{pmatrix} \begin{pmatrix} \psi_- \\ \bar{\lambda}_+ \end{pmatrix} + i(\bar{\psi}_+, \lambda_-) \begin{pmatrix} 2D_z & -\phi \\ \phi^\dagger & -\frac{1}{e^2}\partial_{\bar{z}} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \bar{\lambda}_- \end{pmatrix} \quad (3.62)$$

Using the index theorem of [67], one can see that the operators of the first and the second terms have index 1 and  $-1$  respectively in our background. Furthermore, there is no normalizable zero modes for  $(\bar{\psi}_-, \lambda_+)$  nor  $(\psi_+, \bar{\lambda}_-)$ . To see this we note that

$$\int d^2z \left( |2D_z\psi_+ - \phi\bar{\lambda}_-|^2 + 2e^2|\phi^\dagger\psi_+ - \frac{1}{e^2}\partial_{\bar{z}}\bar{\lambda}_-|^2 \right) \quad (3.63)$$

$$= \int d^2z \left( |2D_z\psi_+|^2 + 2e^2|\phi|^2|\psi_+|^2 + \frac{2}{e^2}|\partial_{\bar{z}}\bar{\lambda}_-|^2 + |\phi|^2|\bar{\lambda}_-|^2 \right) \quad (3.64)$$

in the vortex background, where  $\psi_+$  and  $\bar{\lambda}_-$  are considered here as commuting spinors. The left hand side vanishes for a zero mode of  $(\psi_+, \bar{\lambda}_-)$ , but then vanishing of the right hand side requires  $\psi_+ = \bar{\lambda}_- = 0$ . The argument for  $(\bar{\psi}_-, \lambda_+)$  is the same.

Thus, each of  $(\psi_-, \bar{\lambda}_+)$  and  $(\bar{\psi}_+, \lambda_-)$  has exactly one zero mode. Actually, the expression for the zero mode in terms of the vortex solution is available;

$$\begin{pmatrix} \psi_-^{(0)} \\ \bar{\lambda}_-^{(0)} \end{pmatrix} = \begin{pmatrix} D_z\phi \\ F_{12} \end{pmatrix}, \quad \begin{pmatrix} \bar{\psi}_+^{(0)} \\ \lambda_+^{(0)} \end{pmatrix} = \begin{pmatrix} D_{\bar{z}}\phi^\dagger \\ F_{12} \end{pmatrix}. \quad (3.65)$$

It is easy to verify using the vortex equations (3.53) and (3.54) that these are indeed the zero modes. These come from the supersymmetry transformation under the broken supercharges  $\bar{Q}_-$  and  $Q_+$ . The asymptotic behaviour of  $\phi^\dagger\psi_-^{(0)}$  and  $\bar{\psi}_+^{(0)}\phi$  are

$$\phi^\dagger\psi_-^{(0)} = \partial_z|\phi|^2 = \text{const} \times r_0 \frac{\bar{z}}{|z|} \sqrt{\frac{m}{|z|}} e^{-m|z|} + \dots \quad (3.66)$$

$$\bar{\psi}_+^{(0)}\phi = \partial_{\bar{z}}|\phi|^2 = \text{const} \times r_0 \frac{z}{|z|} \sqrt{\frac{m}{|z|}} e^{-m|z|} + \dots, \quad (3.67)$$

for the vortex at  $z = 0$ . These agree with the asymptotic behaviour of the propagators for a Dirac fermion of mass  $m = e\sqrt{2r_0}$ ;

$$\phi^\dagger \psi_-^{(0)} \propto r_0 S_{--}^F(z), \quad \bar{\psi}_+^{(0)} \phi \propto -r_0 S_{++}^F(z). \quad (3.68)$$

This is not an accident; After a suitable similarity transformation, the operators in (3.62) near infinity (where  $\phi = \sqrt{r}z/|z|$  and  $A_\mu = -\partial_\mu \arg(z)$ ) become nothing but the Dirac operator for a free fermion of mass  $m = e\sqrt{2r}$ .

### *The Computation*

In order to prove the generation of  $e^{-Y}$  term in the twisted superpotential, we would like to compute a quantity that is non-vanishing only when such a term is generated. To find out what is the appropriate object to look at, let us return to our theory of  $\Sigma$  and  $Y$ . We recall that the Lagrangian is

$$\tilde{L} = \int d^4\theta \left( -\frac{1}{2e^2} \bar{\Sigma} \Sigma - \frac{1}{4r_0} \bar{Y} Y \right) + \frac{1}{2} \left( \int d^2\tilde{\theta} \left( \Sigma(Y - t) + c\mu e^{-Y} \right) + c.c. \right), \quad (3.69)$$

and we were asking whether  $c$  is zero or not. Here we have approximated the Kahler potential for  $Y$  by the one in the continuum limit (3.25).

This theory involves a  $U(1)$  gauge field. However, there is no charged field and the only appearance of the gauge field is in the kinetic term and in the Theta term (with the Theta angle being  $\vartheta - \theta$  where  $\vartheta = -\text{Im}(y)$ ). As is well known [68], the effect of the gauge field is to generate a mass term for  $\vartheta$ :

$$U = \frac{e^2}{2} \left( \vartheta - \theta \right)^2, \quad (3.70)$$

where  $(\tilde{\alpha})^2 = \min\{(\alpha + 2\pi n)^2 | n \in \mathbf{Z}\}$ . Thus, we can treat  $\Sigma$  as an ordinary twisted chiral superfield which has a complex auxiliary field. In particular, the theory without  $e^{-Y}$  term is a free theory of two twisted chiral multiplets. It is easy to diagonalize the  $\Sigma Y$  mixing and it turns out that the combinations  $X^{(\pm)} = \pm\Sigma/(2e) + (Y - t)/(2\sqrt{2r_0})$  are superfields of mass

$$\pm m = \pm e\sqrt{2r_0}. \quad (3.71)$$

Now, it is easy to see that the fermionic components  $\chi_+$  and  $\bar{\chi}_-$  of  $\bar{Y}$  has vanishing two point function in the free theory;

$$\langle \chi_+(x) \bar{\chi}_-(y) \rangle = 0, \quad \text{if } c = 0. \quad (3.72)$$

However, if  $e^{-Y}$  is generated, the twisted F-term would contain a term  $-e^{-Y}\overline{\chi}_+\chi_-$  in the Lagrangian. This would contribute to the two point function as

$$\langle \chi_+(x) \overline{\chi}_-(y) \rangle = c\mu r_0^2 \int d^2z e^{-t} S_{++}^F(x-z) S_{--}^F(z-y) \quad (3.73)$$

where  $S_{\alpha\beta}^F(x-y)$  is the Dirac propagator for the fermions of mass  $m = e\sqrt{2r_0}$ .

Now, let us compute the two point function  $\langle \chi_+(x) \overline{\chi}_-(y) \rangle$  in the original gauge theory. We recall from (3.21) that

$$\chi_+ = 2\overline{\psi}_+\phi, \quad \overline{\chi}_- = -2\phi^\dagger\psi_-. \quad (3.74)$$

Since the product of these carries an axial R-charge 2, only the vortex backgrounds with  $k = 1$  can contribute to the two point function. The contribution is expressed as an integration over the location  $z$  of the vortex

$$\langle \chi_+(x) \overline{\chi}_-(y) \rangle = -4\Lambda_{UV} \int d^2z e^{-t_0} (\overline{\psi}_+^{(0)} \phi_{(z)})(x) (\phi_{(z)}^\dagger \psi_-^{(0)})(y), \quad (3.75)$$

where  $\phi_{(z)}$  is the vortex solution at  $z$ . The factor of  $\Lambda_{UV}$  comes from the measure of bosonic zero modes ( $\propto \Lambda_{UV}^2$ ) and that for the fermionic zero modes ( $\propto \Lambda_{UV}^{-1}$ ). As we have seen, the fermionic zero mode multiplied by  $\phi$  is proportional to the Dirac propagator (3.68). Thus, we obtain

$$\langle \chi_+(x) \overline{\chi}_-(y) \rangle = \text{const} \times r_0^2 \Lambda_{UV} \int d^2z e^{-t_0} S_{++}^F(x-z) S_{--}^F(z-y), \quad (3.76)$$

which agrees with (3.73) considering the relation  $\Lambda_{UV} e^{-t_0} = \mu e^{-t}$ . Thus, we have shown that our dual theory correctly reproduces the gauge theory result if and only if  $c \neq 0$ .

### 3.2.3 Solitons and Dualization

In the  $R \rightarrow 1/R$  duality, as reviewed before, the momentum and winding modes get exchanged. This view provides us with another way to interpret the generation of superpotential (3.50). Let us turn off the gauge interaction and consider  $\Sigma$  as a non-dynamical parameter. Before dualization, we have a field  $\Phi$  of mass  $\Sigma$ . In the dual description  $\Phi$  should arise as a winding mode. Indeed if we consider the superpotential

$$W = \Sigma(Y - t) + e^{-Y}. \quad (3.77)$$

Viewing  $\Sigma$  as non-dynamical, the vacua are labeled by  $\partial_Y W = 0$  and we obtain

$$\partial_Y W = 0 \rightarrow e^{-Y} = \Sigma \quad (3.78)$$

and the critical points are given by

$$Y_n = Y_0 + 2n\pi i \quad (3.79)$$

where  $Y_0$  is a special solution of (3.78). The vacua are indexed by an integer  $|n\rangle$ , corresponding to winding number in the  $Y$  plane. This should correspond to momentum mode in the  $\Phi$  variable. In other words the  $\Phi$  field should have  $Y$ -winding number charge  $+1$  and acts on vacua

$$\Phi : |n\rangle \rightarrow |n+1\rangle \quad (3.80)$$

In other words  $\Phi$  should be identified with the soliton interpolating between the vacua. The BPS mass in this sector is given by [69]

$$|m| = \frac{1}{2\pi} |W(Y_n) - W(Y_{n-1})| = |\Sigma|, \quad (3.81)$$

in agreement with the expected mass of  $\Phi$ . This reasoning provides another view point on the superpotential in the actual gauge system. Related ideas were recently discussed in [70].

### 3.3 A Few Generalizations

It is straightforward to extend the description using the dual fields to more general gauge theories. We consider here two generalizations; the case with  $U(1)^k$  gauge group and the case with twisted masses.

#### *Many $U(1)$ Gauge Groups*

The first example is  $U(1)^k$  gauge group with  $N$  matter fields. We denote the field strength superfield for the  $a$ -th gauge group by  $\Sigma_a$  ( $a = 1, \dots, k$ ), and the chiral superfield for the  $i$ -th charged matter by  $\Phi_i$  ( $i = 1, \dots, N$ ). We denote by  $Q_{ia}$  the charge of the  $i$ -th matter under the  $a$ -th gauge group.

The exact twisted superpotential can be obtained by the localization argument which reduces the problem to the sum of copies of the single flavor case. This time, instead of keeping only one gauge coupling we keep  $k$  of them. We start with the sum of  $N$  copies of  $U(1)$  theory with charge 1 matter, and take the weak coupling limit except for the  $U(1)^k$  gauge group embedded in  $U(1)^N$  according to the charge matrix  $Q_{ia}$ . This constrains the  $N$  gauge fields as

$$\Sigma_i = \sum_{a=1}^k Q_{ia} \Sigma_a. \quad (3.82)$$

The exact superpotential for  $\Sigma_a$ 's and the dual  $Y_i$  of  $\Phi_i$  is then given by

$$\widetilde{W} = \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^N Q_{ia} Y_i - t_a \right) + \mu \sum_{i=1}^N e^{-Y_i}. \quad (3.83)$$

Integrating over  $Y_i$ 's, we obtain the following effective twisted superpotential for  $\Sigma_a$ 's:

$$\widetilde{W}_{\text{eff}}(\Sigma_a) = - \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^N Q_{ia} \left( \log \left( \sum_{b=1}^k Q_{ib} \Sigma_b / \mu \right) - 1 \right) + t_a \right). \quad (3.84)$$

This is the effective superpotential that we would obtain if we integrate out  $\Phi_i$ 's in the original gauge theory [65].

### *Twisted Masses*

The next example is  $U(1)^k$  gauge theory with  $N$  charged matter fields as above, but the twisted masses  $\widetilde{m}_i$  for the matter fields are turned on. The twisted mass can be considered as the lowest components of the field strength superfields of the  $U(1)^N/U(1)^k$  flavor symmetry group. This is a non-trivial deformation of our gauge theory. The (anomalous) axial R-symmetry is explicitly broken by this perturbation but is restored if  $\widetilde{m}_i$  are rotated as  $\widetilde{m}_i \mapsto e^{2i\alpha} \widetilde{m}_i$ .

As noted before, the dualization for this case follows from extending the  $U(1)^k$  gauge group to  $U(1)^N$  and taking the suitable decoupling limit. With the twisted masses, the constraint on the field strengths (3.82) is shifted as

$$\Sigma_i = \sum_{a=1}^k Q_{ia} \Sigma_a - \widetilde{m}_i. \quad (3.85)$$

The exact twisted superpotential is thus

$$\widetilde{W} = \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^N Q_{ia} Y_i - t_a \right) + \mu \sum_{i=1}^N e^{-Y_i} - \sum_{i=1}^N \widetilde{m}_i Y_i. \quad (3.86)$$

Note that the net number of deformation parameters is  $(N - k)$  from the flavor group  $U(1)^N/U(1)^k$ ;  $\delta \widetilde{m}_i = \sum_{a=1}^k Q_{ia} c_a$  is absorbed by a shift of the origin of  $\Sigma_a$ 's. Integration over  $Y_i$ 's yields an effective superpotential for  $\Sigma$  that we would obtain if we integrate out  $\Phi_i$ 's in the original gauge theory (as is done in [71] for  $k = 1$  case with  $Q_i = \pm 1$ ).

## **4 Sigma Models From Gauge Theories**

The gauge theory studied in the previous section reduces at low enough energies to the non-linear sigma model on a certain manifold. To see this, we examine the space of

classical vacua of the theory. This can be read by looking at the potential energy for the scalar fields  $\sigma, \phi_i$

$$U = \frac{e^2}{2} \left( \sum_{i=1}^N Q_i |\phi_i|^2 - r_0 \right)^2 + \sum_{i=1}^N Q_i^2 |\sigma|^2 |\phi_i|^2. \quad (4.1)$$

We notice two branches of solutions to the vacuum equation  $U = 0$ ; Higgs branch where  $\sigma = 0$  and  $\sum_{i=1}^N Q_i |\phi_i|^2 = 0$ , and Coulomb branch where  $\sigma$  is free and all  $\phi_i = 0$ . If  $\pm \sum_{i=1}^N Q_i > 0$ ,  $\pm r_0$  is bound to be large and there is only a Higgs branch. If  $\sum_{i=1}^N Q_i = 0$ , there is again only a Higgs branch except  $r_0 \sim 0$  where a Coulomb branch develops. The degrees of freedom transverse to the Higgs branch have masses of order  $e\sqrt{|r_0|}$ , as can be seen from the Lagrangian  $\sum_i |D\phi_i|^2 + (1/2e^2)|d\sigma|^2 + U$ . Thus, for a generic value of the parameter  $r_0$ , the theory at energies much smaller than  $e\sqrt{|r_0|}$  describes the non-linear sigma model on the Higgs branch. This in particular means that all aspects of the sigma model at finite energies (including the BPS soliton spectra for massive sigma models) can be seen by studying the corresponding gauge theory.

To be more precise, the Higgs branch  $X$  is the space of solutions to

$$\sum_{i=1}^N Q_i |\phi_i|^2 = r_0 \quad (4.2)$$

modulo  $U(1)$  gauge transformations  $\phi_i \mapsto e^{iQ_i\gamma}\phi_i$ . This space has complex dimension  $N - 1$  and inherits a structure of Kahler manifold from that of flat  $\mathbf{C}^N$  of  $\phi_i$ 's. It is a standard fact that this is equivalent as a complex manifold to the quotient of  $\mathbf{C}^N - \mathcal{P}$  by the  $\mathbf{C}^\times$  action  $\phi_i \mapsto \lambda^{Q_i}\phi_i$ , where  $\mathcal{P}$  is some subset of codimension  $\geq 1$ . In particular, complex coordinates of  $X$  are represented in the sigma model by the lowest components of chiral superfields. The parameter  $t_0 = r_0 - i\theta$  is identified as the complexified Kahler class. The first Chern class of this space is proportional to  $|\sum_{i=1}^N Q_i|$  and the cut-off dependence (3.3) of  $r_0$  corresponds to the renormalization of the sigma model metric [42]. The sigma model limit is thus

$$e \gg \Lambda \quad (4.3)$$

for  $\sum_{i=1}^N Q_i \neq 0$ .

The space  $X = (\mathbf{C}^N - \mathcal{P})/\mathbf{C}^\times$  has an algebraic torus  $(\mathbf{C}^\times)^{N-1}$  as a group of holomorphic automorphisms; the group  $(\mathbf{C}^\times)^N$  acting on  $\mathbf{C}^N$  in a standard way modulo the complexified gauge group  $\mathbf{C}^\times$ . There is an open subset  $\{\phi_i \neq 0, \forall i\}$  of  $X$  on which  $(\mathbf{C}^\times)^{N-1}$  acts freely and transitively. Such a space  $X$  is called a *toric variety*, or *toric manifold* if it is smooth. As is clear from the equation (4.2), if  $Q_i$  are all positive (or all negative), the manifold  $X$  is compact but if there is a mixture of positive and negative  $Q_i$ 's  $X$  is non-compact.

For  $\sum_{i=1}^N Q_i = 0$ , the FI parameter  $r = r_0$  does not run and  $t = r - i\theta$  is a free parameter of the theory. At  $r = 0$ ,  $X$  contains a singular point  $\phi_i = 0$  where the  $U(1)$  gauge group is unbroken. A new flat direction (Coulomb branch) develops there and the sigma model on  $X$  becomes singular. The actual singularity is determined by the vanishing of the quantum effective potential at large values of  $\sigma$ . Since there is an effective superpotential  $\widetilde{W}_{eff}(\sigma)$  (3.39) which is valid at large  $\sigma$ , this is equivalent to the condition that  $\partial_\sigma \widetilde{W}_{eff}(\sigma) = 0$  at large values of  $\sigma$ . Thus, singularity of the quantum theory is located at

$$t = - \sum_{i=1}^N Q_i \log(Q_i). \quad (4.4)$$

Note that the theory is singular only for a particular value of  $\theta$  (either 0 or  $\pi$ ) and  $r$ . In particular, the theories with  $r \gg 0$  and  $r \ll 0$  are smoothly connected to each other [10], even though the space  $X$  at  $r \gg 0$  differs from  $X$  at  $r \ll 0$ .

One can also consider abelian gauge theory with gauge group  $U(1)^k$  and  $N$  matter fields with  $N \times k$  charge matrix  $Q_{ia}$ . For a generic value of  $t_a$  in a certain range, the vacuum manifold is a Kahler manifold  $X$  of dimension  $N - k$ .  $X$  as a complex manifold is of the type  $(\mathbf{C}^N - \mathcal{P})/(\mathbf{C}^\times)^k$  where  $\mathcal{P}$  is a union of certain planes in  $\mathbf{C}^N$ . The algebraic torus  $(\mathbf{C}^\times)^{N-k} = (\mathbf{C}^\times)^N/(\mathbf{C}^\times)^k$  acts on  $X$  in an obvious way as holomorphic automorphisms and  $X$  contains an open subset on which  $(\mathbf{C}^\times)^{N-k}$  acts freely and transitively. Thus,  $X$  is a toric variety. (In fact, any normal toric variety is obtained this way). We refer the physics reader to [65] for more precise definition of a toric variety and its general properties. See also [72, 73] for more detail. If  $\sum_{i=1}^N Q_{ia} = 0$  for all  $a$ , the theory is parametrized by  $k$  dimensionless parameters  $t_a = r_a - i\theta_a$ . Otherwise, there is a single scale parameter and  $k - 1$  dimensionless parameters. Unlike in the single  $U(1)$  case, the theory can possibly become singular at some locus even if  $\sum_{i=1}^N Q_{ia} \neq 0$ . Singular locus in the parameter space is determined by finding a flat direction in the  $\sigma_a$  space;  $\partial_{\sigma_a} \widetilde{W}_{eff} = 0$  where  $\widetilde{W}_{eff}$  is given in (3.84) (more precisely, we must consider all possible “mixed branches” and do the same computation in the reduced theory). This was studied in detail in [65]. However, as in the single  $U(1)$  case, two generic points in the parameter space can be smoothly connected to each other without meeting a singularity.

The first Chern class  $c_1(X)$  of  $X$  is not necessarily positive semi-definite. In the present paper, we only consider  $X$  with  $c_1(X) \geq 0$ . In such a case, the running of the FI parameter of the gauge theory matches the running of the sigma model coupling at one-loop level, and our gauge theory indeed describes the non-linear sigma model on  $X$  at energies well below the gauge coupling constants. To be more precise, the precise relation of the parameters is possibly complicated when  $c_1(X)$  is close to zero and the one-loop running is not dominant. This can also be understood by comparing the size



of the moduli spaces of gauge theory instantons and instantons of the non-linear sigma model (as shown in [10] in the case of projective hypersurfaces). It was also noted in [74] that the formal parameters corresponding to irrelevant deformations are in complicated relation even when  $c_1$  is large. The precise relation can be determined by finding the so called flat coordinates with the expansion point at infinity. That would lead to the natural coordinates used in the large volume expansion of the non-linear sigma model [75]. (This in particular applies to the mirror QFTs that we will obtain later in this paper in finding the map between the parameters of the non-linear sigma model and the mirror.) If there is a negative component in  $c_1(X)$ , the sigma model is not asymptotic free and is not well-defined. In such a case, our gauge theory has little to do with the manifold  $X$ . If  $c_1(X)$  is negative definite, it is infra-red free and the sigma model (defined as a cut-off theory) flows to a free theory of  $c/3 = \dim X$ .

#### 4.0.1 Examples of Toric Varieties

Let us consider some examples of toric varieties. If we consider a  $U(1)$  gauge theory with  $N$  fields with charges  $+1$ , this gives a linear sigma model realization of  $\mathbf{CP}^{N-1}$ . More generally, if the charges of the matter fields are positive but not necessarily equal, it gives a realization of weighted projective space, with weights determined by the charges.

If we consider a  $U(1)$  gauge theory with  $N$  fields with charge  $+1$  and one with charge  $-d$ , this gives a realization of the total space of the  $\mathcal{O}(-d)$  line bundle over  $\mathbf{CP}^{N-1}$ .

Hirzebruch surface  $F_a$  ( $a = 0, 1, \dots$ ) is a toric manifold of dimension two which is realized as the vacuum manifold of the  $U(1) \times U(1)$  gauge theory with four chiral fields with charges  $(1, 0)$ ,  $(1, 0)$ ,  $(0, 1)$ , and  $(a, 1)$ .  $F_0$  is the product  $\mathbf{CP}^1 \times \mathbf{CP}^1$  and  $F_1$  is the blow up of  $\mathbf{CP}^2$  at one point. The first Chern class is positive for these two cases and it is positive-semi-definite for  $F_2$ . In all these cases, the gauge theory describes the non-linear sigma model. For  $a \geq 3$ , however,  $c_1(F_a)$  is not even positive semi-definite and the sigma model is not well-defined.

#### 4.0.2 Holomorphic Vector Fields and Deformations of Linear Sigma Model

The unitary subgroup  $U(1)^{N-k}$  of the holomorphic automorphism group  $(\mathbf{C}^\times)^{N-k}$  is actually an isometry group of the Kahler manifold  $X$ . Since it is an abelian group, as explained in section 2, we can use this to deform the sigma model on  $X$  by a potential term like (2.26). One may be interested in whether the deformation can be realized in the gauge theory. In fact, this  $U(1)^{N-k}$  is a commutative subgroup of the flavor symmetry

group of the gauge theory, and we can consider the deformation by twisted masses. By construction, the potential deformation of the sigma model is naturally identified as the twisted mass deformation of the gauge group. One can also confirm this by computing the bosonic potential in gauge theory and by checking that it agrees with the potential  $\frac{1}{2}|\widetilde{m}_A V_A|^2 + \frac{1}{2}|\overline{\widetilde{m}}_A V_A|^2$  from the holomorphic isometry (where  $V_A$  are the generators of  $U(1)^{N-k}$ ). We exhibit this in the simplest case  $X = \mathbf{CP}^1$  and leave the general case as an exercise. The  $\mathbf{CP}^1$  sigma model is realized by a  $U(1)$  gauge theory with two fields  $\Phi_1$  and  $\Phi_2$  of unit charge. The potential of the gauge theory with the twisted mass is given by

$$U = \frac{e^2}{2}(|\phi_1|^2 + |\phi_2|^2 - r)^2 + |\sigma - \widetilde{m}|^2|\phi_1|^2 + |\sigma|^2|\phi_2|^2. \quad (4.5)$$

We now fix  $\phi_1$  and  $\phi_2$  to lie in the  $\mathbf{CP}^1$  before perturbation ( $|\phi_1|^2 + |\phi_2|^2 = r$ ) and then extremize the potential with respect to  $\sigma$ . Plugging the result back into (4.5), we obtain the potential

$$U_{\widetilde{m}} = \frac{|\widetilde{m}|^2}{r}|\phi_1|^2|\phi_2|^2. \quad (4.6)$$

On the other hand, the  $U(1)$  isometry generates a holomorphic vector field with  $V^z = iz\partial/\partial z$  where  $z$  is the coordinate of  $\mathbf{CP}^1$  given by  $z = \phi_1/\phi_2$ . The metric of  $\mathbf{CP}^1$  determined by the quotient is the standard Fubini-Study metric  $ds^2 = r|dz|^2/(1 + |z|^2)^2$ . Measuring  $V$  by this, we obtain the potential

$$|\widetilde{m}|^2 \|V\|^2 = |\widetilde{m}|^2 \frac{r|z|^2}{(1 + |z|^2)^2} = |\widetilde{m}|^2 \frac{|\phi_1|^2|\phi_2|^2}{r} \quad (4.7)$$

which is nothing but (4.6).

#### 4.1 Hypersurfaces and Complete Intersections

The non-linear sigma models of hypersurface or complete intersections in a compact toric manifold can also be realized as a gauge theory [10]. Let  $X$  be the compact toric manifold realized as the vacuum manifold of  $U(1)^k$  gauge theory with  $N$  matter fields  $\Phi_i$  of charge matrix  $Q_{ia}$ . We shall consider the submanifold  $M$  of  $X$  defined by the equations

$$G_\beta = 0, \quad \beta = 1, \dots, l, \quad (4.8)$$

where  $G_\beta$  are polynomials of  $\Phi_i$  of charge  $d_{\beta a}$  for the  $a$ -th  $U(1)$  gauge group. Let us add  $l$  matter fields  $P_\beta$  of charge matrix  $-d_{\beta a}$  to the  $U(1)^k$  gauge theory. This theory by itself realizes a non-linear sigma model on a non-compact toric manifold  $V$ .  $V$  is the total space of the sum of  $l$  line bundles on  $X$ ; The new coordinates  $p_\beta$  parametrize the fibre directions and  $X$  is embedded in  $V$  as the zero section  $p_\beta = 0$ . Now let us consider the gauge theory,

with the same gauge group and the same matter content, but having a gauge invariant superpotential of the form

$$W = \sum_{\beta=1}^l P_{\beta} G_{\beta}(\Phi). \quad (4.9)$$

Then, the vacuum equation requires  $G_{\beta} = 0$  and  $\sum_{\beta} p_{\beta} \partial_i G_{\beta} = 0$ . If  $M$  is a smooth complete intersection in  $X$ , this means that  $p_{\beta} = 0$  for all  $\beta$  and the vacuum manifold is  $M$  itself. Thus, the gauge theory with the superpotential (4.9) realizes the non-linear sigma model on the complete intersection  $M$  in  $X$ .

Thus the sigma model on the compact manifold  $M$  is closely related to the sigma model on the non-compact toric manifold  $V$ . In fact they both have the same gauge field and matter content. The only difference between them is that, in the compact theory, there is an F-term involving a gauge invariant superpotential  $W$  which yields *cc* ring deformation. In this sense the compact theory can be embedded in the non-compact theory. The non-compact theory is obtained by considering  $W \rightarrow \epsilon W$  in the limit of setting  $\epsilon \rightarrow 0$ . Of course this limit changes drastically the behavior of the theory and in particular the theory has  $2l$  more complex dimensions in the UV.

However, we can ask if there are any quantities which are unaffected by this deformation. The answer is that if we are considering quantities (such as *ac* ring) which are sensitive only to the twisted F-terms such as Kahler parameters (the FI terms of the gauge theory), then they should not depend on the F-term deformations. Thus for those questions it should be irrelevant whether we are considering the compact theory or the non-compact theory. All that we have to do is to find out how the states and the operators of the compact theory are embedded in that of the non-compact theory and compute the protected quantities that way. This is in fact analogous to embedding questions of vacuum geometry from one theory in a theory of higher central charge, discussed in [26]. It is also similar to the computation of the elliptic genus of minimal models using a free theory with higher central charge [76].

In the present case this issue has been studied in [65] where they also obtain the embedding of the chiral field of the sigma model on  $M$  theory with that of  $V$ . This result is stated as follows. Let  $\delta_{\beta}$  ( $\beta = 1, \dots, l$ ) be the *ac* ring element of the theory on  $V$  defined by

$$\delta_{\beta} = \sum_{a=1}^k d_{\beta a} \Sigma_a, \quad (4.10)$$

where  $\Sigma_a$  is the field strength of the  $a$ -th gauge group. Then, the correlation functions  $\langle \dots \rangle_M$  of the A-twisted model on  $M$  are obtained from those  $\langle \dots \rangle_V$  on  $V$  by

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_M = \langle \mathcal{O}_1 \cdots \mathcal{O}_s(-\delta_1^2) \cdots (-\delta_l^2) \rangle_V. \quad (4.11)$$

An intuitive understanding of this relation is available in the quantum mechanics obtained by dimensional reduction. Let  $L_a$  be a line bundle over  $V$  defined by the equivalence relation  $(p_\beta, \phi_i; c) \equiv (\prod_a \lambda_a^{-d_{\beta a}} p_\beta, \prod_a \lambda_a^{Q_{ia}} \phi_i; \lambda_a c)$  where  $(\lambda_a)$  is the element of the complexified gauge group  $(\mathbf{C}^\times)^k$  and  $c$  is the fibre coordinate. Then, one can show that the  $ac$  ring element  $\Sigma_a$  corresponds to the first Chern class  $c_1(L_a)$  of  $L_a$ . Now, let us consider the tensor product

$$L_\beta = \bigotimes_{a=1}^k L_a^{\otimes d_{\beta a}}, \quad (4.12)$$

whose first Chern class corresponds to the  $ac$  ring element  $\delta_\beta$ . The line bundle  $L_\beta^{-1}$  has a section proportional to  $p_\beta$  and thus  $-\delta_\beta$  represents the divisor class of  $p_\beta = 0$  in  $V$ . In particular, the product  $(-\delta_1) \cdots (-\delta_l)$  represents the class  $X$  in  $V$ . On the other hand, the bundle  $L_\beta$  has a section proportional to  $G_\beta$  and thus  $\delta_\beta$  represents the divisor class of  $G_\beta = 0$ . Therefore,  $(-\delta_1^2) \cdots (-\delta_l^2)$  represents a delta function supported on  $M$ . What is shown in [65] is basically that this quantum mechanical interpretation remains true for the full 2d QFT as long as topological correlators are concerned. It turns out that we need a stronger version of the relation (4.11). We thus proceed to a physical derivation of this relation, which uses the fact that the compact and non-compact theories are embedded in the same underlying physical theory, and yields the stronger version that we need.

The basic idea is that with  $\epsilon \neq 0$  we have turned on a superpotential and we can follow the states from the non-compact theory to the compact theory. In this sense the non-compact theory flows in the IR limit (for non-vanishing  $\epsilon$ ) to the compact theory. We can thus follow the states in the non-compact theory and ask which ones survive in the IR limit. By the nature of the RG flow, we will be losing some states as we take the IR limit. However, if we concentrate on the ground states in the Ramond sector which correspond to normalizable states in the non-compact theory, then in the compact theory they are bound to survive as a ground state (the same cannot be said of the non-normalizable ground states in the non-compact theory, which might disappear from the spectrum of the compact theory).

In the sigma model on the non-compact manifold  $V$ , the ground state corresponding to the operator  $\delta = \delta_1 \cdots \delta_l$  is a normalizable state which is the product of Kahler forms that control the sizes of the compact part of the geometry. We denote this state by

$$|\delta\rangle_V. \quad (4.13)$$

In the large volume limit, this state has the axial R-charge  $-\dim_{\mathbf{C}} V + 2l = -\dim_{\mathbf{C}} M$  which is the lowest among normalizable ground states. Now, let us turn on the superpotential  $\epsilon W$  where  $W$  is the one given in (4.9). The state  $|\delta\rangle_V$  which is the unique

normalizable ground state in the Ramond sector with axial charge  $-\dim_{\mathbf{C}} V + 2l$  will be deformed to a state which we denote by  $|\delta\rangle_{\epsilon}$ . By standard supersymmetry arguments, this state is a normalizable ground state of the Ramond sector. For large  $\epsilon$  (or for any finite  $\epsilon$  in the IR limit) the theory corresponds to the sigma model on the compact manifold  $M$  for which there is a unique ground state with axial charge  $-\dim_{\mathbf{C}} M$ , which is also sometimes denoted by  $|1\rangle_M$ . Thus we can identify  $|\delta\rangle_{\epsilon} = |1\rangle_M$  as long as axial charge is conserved. Thus, we have the following correspondence of states as we increase  $\epsilon$ ;

$$|\delta\rangle_V \xrightarrow{\epsilon \neq 0} |\delta\rangle_{\epsilon} \xrightarrow{\epsilon \rightarrow \infty} |1\rangle_M. \quad (4.14)$$

Here we have used the axial R-charge to identify the state to which  $|\delta\rangle_V$  is deformed. For the more general case, where the axial R-symmetry is broken by an anomaly, the result still remains true, as we will now argue. To see this, note that at large Kahler parameters, i.e. as  $t_a \rightarrow \infty$  axial R-charge is a good symmetry. So at least to leading order it is correct. To show it is true for all  $t_a$  we proceed as follows: According to [57] the topologically twisted theory picks a section of the vacuum bundle which is holomorphic in the sense defined by the topological twisting. In particular we can choose the perturbed state  $|\delta\rangle_{\epsilon}$  so that

$$\frac{\partial}{\partial t_a} |\delta\rangle_{\epsilon} = 0. \quad (4.15)$$

For infinitely large  $t_a$ , since the axial charge is conserved the state  $|\delta\rangle_V$  is deformed for finite  $\epsilon$  to  $|1\rangle_M$ . For finite but large  $t_a$  the fact that we can choose a holomorphic section of the vacuum bundle shows that the difference between  $|\delta\rangle_{\epsilon}$  and  $|1\rangle_M$  can only be given by states with coefficients involving some powers of  $q_a = e^{-t_a}$ . Here we note that the axial R-charge can be made conserved by shifting the Theta angle to cancel (3.4). This in particular means that we assign *non-negative* R-charges to  $q_a$  as long as the first Chern class of  $M$  is positive-semi-definite. It thus follows that  $q_a$ 's would be accompanied by states which will have too small an R-charge to correspond to a normalizable state, since  $|\delta\rangle_V$  is the unique normalizable ground state with minimum R-charge. This establishes the relation (4.14) in general.

The relation (4.14) can be used to yield another derivation of the relation (4.11): The topological correlations for sigma model on  $M$  can be written as

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_M = \langle 1 | \mathcal{O}_1 \cdots \mathcal{O}_s | 1 \rangle_M = \langle \delta | \mathcal{O}_1 \cdots \mathcal{O}_s | \delta \rangle_V = \langle (-\delta_1^2) \cdots (-\delta_l^2) \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_V. \quad (4.16)$$

where we have used the  $\epsilon$  independence in the topological A-model computations, as  $\epsilon W$  does not affect the *ac* correlation functions (note that the overall proportionality factor is not fixed, and depends on the choice of normalization of the topologically twisted theory).

### 4.1.1 Examples of Hypersurfaces

Consider a  $U(1)$  gauge theory with  $N$  fields  $\Phi_i$  of charge  $+1$  and one field  $P$  with charge  $-d$ . So far this is the same as the linear sigma model description of the total space of the line bundle  $\mathcal{O}(-d)$  over  $\mathbf{CP}^{N-1}$ . However, now consider adding to the action the F-term with superpotential

$$W = PG(\Phi_i) \quad (4.17)$$

where  $G$  is a polynomial of degree  $d$  in  $\Phi_i$ . This gauge theory describes, in the infrared, the sigma model on a hypersurface of degree  $d$  in  $\mathbf{CP}^{N-1}$ . The hypersurface for  $d = N$  is a Calabi-Yau manifold of complex dimension  $N - 2$ . For  $d < n$  it is a manifold of positive  $c_1$  and the sigma model is asymptotically free. For  $d > n$  it is a manifold with negative  $c_1$  and the sigma model is not asymptotically free.

The geometric regime corresponds to when the FI parameter is very large  $r \gg 0$  (which is the UV limit for  $d < N$ ). If we consider the limit  $r \rightarrow -\infty$  (which is the IR limit for  $d < N$ ), we end up with an LG theory [10]:  $P$  picks up an expectation value and breaks the  $U(1)$  gauge symmetry to a discrete subgroup  $\mathbf{Z}_d$  and the effective theory is given by the LG model with  $W = G(\Phi_i)$  divided by a  $\mathbf{Z}_d$  action, which corresponds to multiplying the fields  $\Phi_i$  by a  $d$ -th root of unity. If we turn off the superpotential, i.e., back in the non-compact toric model corresponding to the total space of the line bundle  $\mathcal{O}(-d)$  over  $\mathbf{CP}^{N-1}$  the limit  $r \rightarrow -\infty$  would correspond to the orbifold of the free theory of  $N$  fields  $\Phi_i$  modded out by  $\mathbf{Z}_d$ , acting as the  $d$ -th root of unity on  $\Phi_i$ .

The generalization of the above to the case of hypersurfaces of weighted projective space are straightforward.

## 5 A Proof Of Mirror Symmetry

In this section we show how the results of section 3 on the dynamics of supersymmetric gauge theories naturally lead to a proof of mirror symmetry. Before embarking on the proof, we discuss what we mean by the “proof”.

### 5.1 What We Mean by Proof of Mirror Symmetry

As we have discussed before the action for a  $(2, 2)$  supersymmetric field theory has three types of terms: D-terms, F-terms and twisted F-terms. In the case of a supersymmetric sigma model, these three types of terms have the following interpretations: The D-terms correspond to changing the metric on the target manifold without changing the Kahler or

complex parameters. For example for  $\mathbf{CP}^1$  we can consider a fixed total area, but deform the metric from the round metric to any other metric with the same total area by varying the D-term. Variations of F-terms and twisted F-terms correspond to deformations of complex and Kahler parameters of the target space. As discussed before, F-terms and twisted F-terms control many important aspects of the  $(2, 2)$  theories. In particular the  $cc$  and  $ac$  rings depend only on F-terms and twisted F-terms respectively and are independent of D-terms. These in particular encode the instanton corrections of the sigma model to the  $ac$  ring. Also BPS structure of the massive  $(2, 2)$  theories are completely determined by the F-terms and twisted F-terms [57, 26, 78]. One can define the notion of D-branes for  $(2, 2)$  theories (as is familiar in the conformal case, and can be easily extended to the non-conformal case as we will discuss in section 6). One can also show that the overlap of the corresponding boundary states with vacua will only depend on the F-terms (or twisted F-terms depending on which combination of supercharge the D-brane preserves).

These indicate the importance of F-terms and twisted F-terms for the  $(2, 2)$  theories. In fact they become even more prominent in the conformal case, for example for the case of Calabi-Yau manifolds. Namely in that case the sigma model flows in the infrared limit to a conformal field theory which is determined entirely by the complex and Kahler parameters of the manifold. In other words the metric on the Calabi-Yau manifold adjusts itself to the unique form consistent with conformal invariance for a given Kahler and complex structure. In particular the D-terms are entirely *determined* by the F-terms and twisted F-terms in this case, as far as the IR behavior is concerned. As a consequence, a deformation of the conformal field theory corresponds to a deformation of the Kahler or complex structure of the Calabi-Yau manifold, which in turn is realized through variations of F-terms and twisted F-terms. From these facts, and from the fact that the  $cc$  and  $ac$  rings are determined in terms of F-terms and twisted F-terms one wonders whether the opposite is true (i.e. whether the  $cc$  and  $ac$  rings determine the F-terms and twisted F-terms and thus the full theory). This is, however, not completely true; theories with the same chiral ring can differ in the “integral structure” [77]. More precisely, in order to completely specify the theory we need to know the structure of the allowed D-branes.

We prove mirror symmetry in two different senses: In the strong sense, we find a dual theory, for which we prove it is equivalent to the original theory, up to deformations involving D-terms. In the weak sense, we propose a dual theory for which we can only show that the  $cc$ ,  $ac$ , BPS structure of solitons and the  $D$ -brane structure are the same. The weak and strong senses become equivalent if we assume that the  $cc$  and  $ac$  ring, together with the integral structure determine the theory up to D-term variations.

We are now ready to discuss the proof for mirror symmetry. We divide the discussion

into several cases and indicate in each case what is the sense of the proof (strong/weak) that we shall present.

## 5.2 The Proof

The cases of interest naturally divide into three different classes:

- (i)  $X = \mathbf{CP}^{N-1}$  or a more general compact toric manifold with  $c_1(X) \geq 0$ ,
- (ii)  $X$  is a non-compact Calabi-Yau or a more general non-compact toric manifold.

And finally,

- (iii) Sigma models on hypersurfaces or complete intersection in  $X$ .

It turns out that the mirror for the cases (i) and (ii) are rather straightforward to derive, using the tools we have developed so far and we shall prove them in the strong sense. For the derivation of the case (iii) we need an additional tool, which we will discuss in section 6, and we postpone a complete discussion of (iii) to section 7. This will lead to a proof of case (iii) in the weak sense. In this section after deriving the general mirror for the cases (i) and (ii) we briefly mention how it effectively gives an answer of the case (iii) (to be more fully developed in section 7). We present some examples for cases (i) and (ii) and how it relates to case (iii) to illustrate the meaning of the results.

We first consider the sigma model on a toric manifold which is described by a gauge theory with a single  $U(1)$  gauge group with chiral fields of charge  $Q_i$  ( $i = 1, \dots, N$ ). We recall that the dual of the gauge theory is described by a vector multiplet with field strength  $\Sigma$  and  $N$  periodic variable  $Y_i$  dual to the charged matter fields. It has an approximate Kahler potential  $(1/2e^2)|\Sigma|^2 + \sum_i |Y_i|^2/4r_0$  and an exact twisted superpotential

$$\widetilde{W} = \Sigma \left( \sum_{i=1}^N Q_i Y_i - t(\mu) \right) + \mu \sum_{i=1}^N e^{-Y_i}. \quad (5.1)$$

The fields  $\Sigma$  and  $\sum_i Q_i Y_i$  have mass of order  $e\sqrt{|r_0|}$  while the mass scale for the modes tangent to  $\sum_i Q_i Y_i = t$  is of order  $\mu\sqrt{|r_0|}$  where  $\mu \sim \Lambda$  for  $\sum_i Q_i \neq 0$ . In the sigma model limit  $e \gg \mu$ , these mass scales are well-separated and it becomes appropriate to integrate out  $\Sigma$ . This yields the constraint

$$\sum_{i=1}^N Q_i Y_i = t. \quad (5.2)$$

Clearly this can be solved by  $N - 1$  periodic variables. Now, the dual theory becomes the



theory of such  $N - 1$  variables solving (5.2) with the twisted superpotential

$$\widetilde{W} = \mu \sum_{i=1}^N e^{-Y_i}. \quad (5.3)$$

This can be interpreted as a sigma model on  $(\mathbf{C}^\times)^{N-1}$  with a twisted superpotential. Since the complex coordinates are the lowest components of twisted chiral superfields, this theory can be identified as the mirror of the non-linear sigma model on  $X$ .

It is straightforward to extend this to the more general case where we start with  $k$   $U(1)$  gauge groups and  $N$  matter fields of charge  $Q_{ia}$  ( $i = 1, \dots, N$ ,  $a = 1, \dots, k$ ). The target space  $X$  of the sigma model is the quotient of

$$\sum_{i=1}^N Q_{ia} |\phi_i|^2 = r_a \quad (5.4)$$

by the  $U(1)^k$  action  $\phi_i \mapsto e^{iQ_{ia}\gamma_a} \phi_i$ , which is a toric variety of dimension  $N - k$ . The dual description of the gauge theory is obtained in (3.83). Integrating out the vector multiplet, we obtain the algebraic torus  $(\mathbf{C}^\times)^{N-k}$  as the solutions to

$$\sum_{i=1}^N Q_{ia} Y_i = t_a. \quad (5.5)$$

The dual theory is a sigma model on this  $(\mathbf{C}^\times)^{N-k}$  with the twisted superpotential

$$\widetilde{W} = \mu \sum_{i=1}^N e^{-Y_i}. \quad (5.6)$$

This can be identified as the mirror of the sigma model on  $X$ . We thus have established the mirror symmetry of the non-linear sigma model on a toric variety  $X^{N-k}$  and the theory on the algebraic torus  $(\mathbf{C}^\times)^{N-k}$  with a superpotential.

We can also consider deforming the sigma model using holomorphic vector fields, as discussed in section 2. There are  $(N - k)$  such parameters corresponding to the  $U(1)^N/U(1)^k$  holomorphic isometry group for the above toric variety. We parameterize these deformations by  $N$  parameters  $\widetilde{m}_i$  (as we will see below  $k$  of them are redundant). The corresponding gauge theory is the one with the twisted masses  $\widetilde{m}_i$  and the twisted superpotential for the dual theory is obtained in (3.86). Adding the twisted masses does not affect the elimination of the field strength  $\Sigma_a$  and we obtain the same constraints (5.5). All it does is to shift the superpotential (5.6) as

$$\widetilde{W} = \mu \sum_{i=1}^N e^{-Y_i} - \sum_{i=1}^N \widetilde{m}_i Y_i. \quad (5.7)$$

This is not a single valued function on  $(\mathbf{C}^\times)^{N-k}$ , but the multivaluedness is a constant shift and therefore the Lagrangian itself is well-defined. It is easy to see, using (5.5) that  $k$  of the parameters  $\widetilde{m}_i$  are redundant and do not affect the superpotential.

We will now illustrate these ideas in two concrete cases: One is in the context of compact toric varieties which we exemplify by using the case of  $\mathbf{CP}^{N-1}$ . The second one is for the case of non-compact toric varieties which we exemplify by considering the total space of  $\mathcal{O}(-d)$  line bundle over  $\mathbf{CP}^{N-1}$ . In the context of the latter example we also explain briefly how the mirror for the hypersurface case arises. We complete the discussion for the hypersurface case in section 7.

### 5.3 Compact Toric Manifold

#### *The $\mathbf{CP}^{N-1}$ Model*

The linear sigma model for  $X = \mathbf{CP}^{N-1}$  is the  $U(1)$  gauge theory with  $N$  matter fields of charge 1. The constraint  $\sum_{i=1}^N Y_i = t$  is solved by  $Y_i = t/N - \Theta_i$  ( $i = 1, \dots, N-1$ ) and  $Y_N = t/N + \sum_{i=1}^{N-1} \Theta_i$ , where  $\Theta_i$  are periodic variables of period  $2\pi i$  and can be considered as coordinates of  $(\mathbf{C}^\times)^{N-1}$ . The superpotential is

$$\widetilde{W} = \Lambda \left( e^{\Theta_1} + \dots + e^{\Theta_{N-1}} + \prod_{i=1}^{N-1} e^{-\Theta_i} \right), \quad (5.8)$$

where  $\Lambda = \mu e^{-t/N}$  is the dynamical scale of the theory. This is the superpotential for supersymmetric affine  $A_{N-1}$  Toda field theory. Thus, we have derived the mirror symmetry of  $\mathbf{CP}^{N-1}$  model and affine Toda theory which was observed in [25, 26, 16, 28, 29] from various points of view.

Having no F-term, the theory is invariant under  $U(1)_V$  R-symmetry. The twisted superpotential (5.8) explicitly breaks  $U(1)_A$  but its  $\mathbf{Z}_{2N}$  subgroup remains unbroken; for  $\Theta_j \rightarrow \Theta_j + 2\pi i/N$ ,  $\widetilde{W} \rightarrow e^{2\pi i/N} \widetilde{W}$ . The vacua of the theory are given by the critical points of  $\widetilde{W}$ ,  $\partial_{\Theta_i} \widetilde{W} = 0$ . It is solved by  $e^{\Theta_1} = \dots = e^{\Theta_{N-1}} =: X$  where  $X = X^{-(N-1)}$ . Namely, there are  $N$  vacua at  $e^{\Theta_j} = e^{2\pi i \ell/N}$  (all  $j$ ) with the critical value

$$\widetilde{W} = N\Lambda e^{2\pi i \ell/N}, \quad (5.9)$$

( $\ell = 0, \dots, N-1$ ). Each vacuum is massive and breaks spontaneously the axial R-symmetry  $\mathbf{Z}_{2N}$  to  $\mathbf{Z}_2$ . All these are indeed the properties which are possessed by the  $\mathbf{CP}^{N-1}$  model.

An advantage of LG type description is that the BPS soliton spectrum can be exactly analyzed using the superpotential. A BPS soliton corresponds to a trajectory connecting

two vacua which projects onto a straight line in the  $\widetilde{W}$  space. The mass is equal to the absolute value of the susy central charge which is given by the difference of the two critical values of  $\widetilde{W}$ . From (5.9) we see that the BPS soliton connecting the 0-th and the  $\ell$ -th vacua has central charge

$$\widetilde{Z}_{0\ell} = N\Lambda(e^{2\pi i\ell/N} - 1). \quad (5.10)$$

One can also see that there are  $\binom{N}{\ell}$  such solitons [79]. For each of them,  $\ell$  of  $e^{-Y_i}$  are equal to a trajectory  $f_\ell$  in  $\mathbf{C}^\times$  while the remaining  $(N-\ell)$  of them to another  $f_{N-\ell}$ . They both starts from  $e^{-t/N}$  and ends at  $e^{-t/N+2\pi i\ell/N}$  but  $f_\ell$  has a relative winding number  $(-1)$  compared to  $f_{N-\ell}$ . Since the winding in  $Y_i$  is dual to the charge for phase rotation of  $\Phi_i$ , the soliton has the same quantum number as the product  $\Phi_{i_1} \cdots \Phi_{i_\ell}$ . Indeed, the soliton spectrum of the  $\mathbf{CP}^{N-1}$  model has been studied in [61] and it was found that the  $\ell = 1$  solitons are the elementary electrons  $\Phi_i$  and the higher- $\ell$  solitons are their  $\ell$ -th antisymmetric products. This spectrum for the solitons was also recovered from the  $tt^*$  geometry in [80].

We can also consider deforming the  $\mathbf{CP}^{N-1}$  sigma model by the addition of a combination of the holomorphic vector fields  $U(1)^{N-1}$ . In such a case we obtain the mirror (5.7):

$$\widetilde{W} = \Lambda \left( e^{\Theta_1} + \cdots + e^{\Theta_{N-1}} + \prod_{i=1}^{N-1} e^{-\Theta_i} \right) + \sum_{i=1}^{N-1} (\widetilde{m}_i - \widetilde{m}_N) \Theta_i, \quad (5.11)$$

As has been noted,  $\widetilde{m}_i$ 's changes the central charge of the supersymmetry algebra by a term proportional to the charges  $S_i$  of the global abelian symmetry  $U(1)^{N-1}$ ,

$$\delta \widetilde{Z} = \sum_{i=1}^{N-1} (\widetilde{m}_i - \widetilde{m}_N) S_i. \quad (5.12)$$

The charges  $S_i$  can be identified with the weights of the  $SU(N)$  global symmetry. In this way we can recover not only the soliton spectrum, but also their quantum numbers under the  $SU(N)$  global symmetry, and obtain the anticipated result noted above [79]. Note also, that the above deformation, deforms the quantum cohomology ring. If we denote by  $x = -d\widetilde{W}/dt$  the generator of the chiral ring (corresponding to the Kahler class of  $\mathbf{CP}^{N-1}$ ), from the above superpotential one obtains

$$\prod_{i=1}^N (x - \widetilde{m}_i) = \Lambda^N. \quad (5.13)$$

Note that the twisted masses deforms the cohomology ring from the simple form  $x^N = \Lambda^N$  to an arbitrary polynomial of degree  $N$ . The result for the case of  $\mathbf{CP}^1$  was first derived through other arguments in [81]. The general case was conjectured in [71] from brane construction of the theory.

As noted before, we do not attempt to specify the D-terms which are subject to perturbative corrections in general. However it is expected that the sine-Gordon theory ( $A_1$  affine Toda theory) is integrable and the D-term is protected from quantum correction [82, 25]. Thus, one may expect that a more detailed statement of the mirror symmetry can be made. The  $\mathbf{CP}^1$  model realized as the linear sigma model possesses the  $SU(2)$  isometry group as the global symmetry. Recall that we have seen that the Kahler potential for  $Y_i$ 's is approximately  $|Y_i|^2/4r_0$  at the classical level. Recall also that  $r_0 \rightarrow \infty$  in the continuum limit. This suggests that the equivalence of the  $SU(2)$  invariant  $\mathbf{CP}^1$  model and the sine-Gordon theory holds only in the limit where the Kahler potential of the latter vanishes. This is actually consistent with the observation [82] that the  $N = 2$  sine-Gordon theory possesses  $SU(2)$  global symmetry, rather than its q-deformation, in the limit of vanishing Kahler potential. Also, it was observed in [25] that the scattering matrix of the BPS solitons of the  $\mathbf{CP}^1$  model and the sine-Gordon theory agree with each other in such a limit. It would be interesting to investigate the integrability and the protection of the D-term in more general cases.

### *More General Cases*

We now study a general aspects of the mirror theory for more general toric manifold  $X$ . Let us consider the equations  $\sum_{i=1}^N v_i Q_{ia} = 0$  for integers  $v_i$ . The space of solutions form a lattice of rank  $N - k$ . Let  $v_i^A$  be the integral basis of this lattice;

$$\sum_{i=1}^N v_i^A Q_{ia} = 0, \quad \begin{cases} A = 1, \dots, N - k, \\ a = 1, \dots, k. \end{cases} \quad (5.14)$$

The constraints (5.5) on  $Y_i$  can be solved by  $N - k$  periodic variables  $\Theta_A$  as

$$Y_i = \sum_{A=1}^{N-k} v_i^A \Theta_A + t_i \quad (5.15)$$

where  $(t_i)$  is a solution to  $\sum_{i=1}^N Q_{ia} t_i = t_a$  (an arbitrary choice will do; another choice is related by a shift of  $\Theta_A$ 's). Now, the superpotential (5.6) of the mirror theory can be expressed as

$$\widetilde{W} = \sum_{i=1}^N \exp(-t_i - \langle \Theta, v_i \rangle), \quad (5.16)$$

where  $\langle \Theta, v_i \rangle$  is the short hand notation for  $\sum_{A=1}^{N-k} v_i^A \Theta_A$ . Note that the expression (5.16) is the same as the function in [83] which determines the quantum cohomology of toric manifolds. Namely, we have derived the result of [83] as a straightforward consequence of our dual description.

It is useful to consider  $v_i = (v_i^A)$  as (generically linearly dependent)  $N$  vectors in the lattice  $N = \mathbf{Z}^{N-k}$  and  $\Theta_A$  as the coordinates on  $M_{\mathbf{C}} = M \otimes \mathbf{C}$  where  $M$  is the dual lattice of  $N$ . Then,  $\langle \Theta, v_i \rangle$  that appears in (5.16) can be identified as the natural pairing. It is well-known in toric geometry [72, 73] that  $X$  is compact if and only if the cone generated by  $v_i$  covers the whole  $N_{\mathbf{R}} = N \otimes \mathbf{R}$  (i.e. any element of  $N_{\mathbf{R}}$  is expressed as a linear combination of  $v_i$ 's with non-negative coefficients). This in particular means that there is no value of  $\Theta \in M_{\mathbf{C}}$  such that  $\text{Re}\langle \Theta, v_i \rangle \geq 0$  for all  $i$ . In other words, for any non-zero  $\text{Re}\Theta$ ,  $\text{Re}\langle \Theta, v_i \rangle < 0$  for some  $i$ . Thus, there is no obvious run-away direction of the superpotential (5.16) and we generically expect a discrete spectrum.<sup>1</sup> This is of course an expected property for a mirror of a compact sigma model.

Since the Witten index of the sigma model is equal to the Euler number

$$\text{Tr}(-1)^F = \chi(X), \quad (5.17)$$

the number of critical points of the superpotential (5.16) must agree with  $\chi(X)$ . To check this, we note another well-known fact in toric geometry: Our algebraic torus  $(\mathbf{C}^\times)^{N-k}$  (with coordinates  $e^{\Theta_A}$ ) is embedded as an open subset of a “dual” toric variety  $Y$  and each term in (5.16) extends to a section of the anti-canonical bundle  $K_Y^{-1}$  of  $Y$ .<sup>2</sup> Thus, each partial derivative  $\partial_A \widetilde{W} = \partial \widetilde{W} / \partial \Theta_A$  also extends to a section  $s_A$  of  $K_Y^{-1}$ . Since a critical point is a common zero of all  $\partial_A \widetilde{W}$ 's, the number of critical points is the number of intersection points of the divisors  $\partial_A \widetilde{W} = 0$  in  $(\mathbf{C}^\times)^{N-k}$ . Since there is no run-away behaviour of the potential for generic values of  $t_a$ , we do not expect a common zero of  $s_A$ 's at infinity  $Y - (\mathbf{C}^\times)^{N-k}$ . Thus, the number of critical points must be the same as the topological intersection number of the  $N - k$  divisors  $s_A = 0$  in  $Y$  which is counted as

$$\text{Tr}_{\text{mirror}}(-1)^F = \langle [Y], c_1(K_Y^{-1})^{N-k} \rangle. \quad (5.18)$$

Therefore the number (5.18) must agree with the Euler number of  $X$ . It appears that this is not known in general. However, this certainly holds in every example one can check as long as  $c_1(X) \geq 0$ .

Since the sigma model is well-defined and the gauge theory agrees with it only when the first Chern class of  $X$  is positive semi-definite, the above result makes sense only in

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<sup>1</sup>There can be an accidental situation (which appears in a sublocus of the parameter space) where there is a run-away direction. This comes from the singularity of the sigma model that is associated with the development of the Coulomb or mixed branch, which is studied in [65].

<sup>2</sup>The “dual” toric variety is constructed as follows. Take the convex hull of  $v_i$ 's in  $N_{\mathbf{R}}$ ; this makes a convex polytope  $\Delta$  in  $N_{\mathbf{R}}$ . Consider the dual  $\Delta^\circ \subset M_{\mathbf{R}}$  of  $\Delta \subset N_{\mathbf{R}}$  defined as the set of points in  $M_{\mathbf{R}}$  whose values at  $\Delta$  are  $\geq -1$ . Then, the vertices  $\{u_I\}$  of  $\Delta^\circ$  determines the “dual” toric variety  $Y$ .

such cases  $c_1(X) \geq 0$ . To illustrate this, we consider the Hirzebruch surface  $F_a$ . As noted before, the charge matrix for  $F_a$  is  $Q_{i1} = {}^t(1, 0, 1, a)$  and  $Q_{i2} = {}^t(0, 1, 0, 1)$  and therefore the vectors  $v_i^A$  are given by  $v_1 = (1, 0)$ ,  $v_2 = (0, 1)$ ,  $v_3 = (-1, a)$  and  $v_4 = (0, -1)$ . Then the superpotential is given by

$$\widetilde{W} = e^{-\Theta_1} + e^{-t_2}e^{-\Theta_2} + e^{-t_1}e^{\Theta_1 - a\Theta_2} + e^{\Theta_2}. \quad (5.19)$$

It is easy to see that the number of critical points are 4 for  $a = 0, 1$  and  $2a$  for  $a \geq 2$ . Since the Euler number of  $F_a$  is always 4, the result is “correct” only for  $a = 0, 1, 2$ . This is consistent because  $c_1(F_a)$  is positive semi-definite only for  $a = 0, 1, 2$ .

#### 5.4 Non-Compact Case

In the case where  $X$  is non-compact there is an obvious run-away direction of the superpotential  $\widetilde{W}$ . To see this, we note that  $X$  is non-compact if and only if  $v_i$ ’s generate a proper convex cone in  $\mathbf{N}_{\mathbf{R}}$ . This means that there is a point  $\Theta_0$  in  $\mathbf{M}_{\mathbf{R}}$  such that  $\langle \Theta_0, v_i \rangle \geq 0$  for all  $i$ . Now, consider the behaviour of the superpotential  $\widetilde{W}$  at  $\Theta = t\Theta_0$  in the limit  $t \rightarrow +\infty$ . In the case where  $\langle \Theta_0, v_i \rangle > 0$  for all  $i$ , each term of the superpotential vanishes in the limit and this is the run-away direction. If there is some  $i$  such that  $\langle \Theta_0, v_i \rangle = 0$  the superpotential stays finite but can be extremized by choosing an appropriate  $\Theta_0$ . In any case, there is a run-away direction of the superpotential. We thus expect a continuous spectrum, which is indeed a property of the sigma model on a non-compact manifold.

Below, we present two basic examples of non-compact toric manifolds. The first one, the total space of  $\mathcal{O}(-d)$  over  $\mathbf{CP}^{N-1}$ , provides a starting point of the discussion of the mirror for hypersurfaces in  $\mathbf{CP}^{N-1}$  and more general toric complete intersections. The second one, the total space of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$  over  $\mathbf{CP}^1$ , is important for the study of phase transition in the sense of [10, 84].

$\mathcal{O}(-d)$  over  $\mathbf{CP}^{N-1}$

The linear sigma model for the non-compact space given by the total space  $\mathcal{O}(-d)$  over  $\mathbf{CP}^{N-1}$  is given by a  $U(1)$  gauge theory with  $N$  fields with charge  $+1$  and one field with charge  $-d$ . As discussed before we find that the mirror for this theory is an LG theory given by an LG theory with superpotential

$$W = \sum_{i=1}^N e^{-Y_i} + e^{-Y_P}$$

where

$$-dY_P + \sum_{i=1}^N Y_i = t \quad (5.20)$$

and  $Y_P$  is the dual field to the charge  $-d$  matter and  $Y_i$  for  $i = 1, \dots, N$  are dual to the charge  $+1$  matter fields. It is natural to use (5.20) to solve for  $Y_P$ . Let us define

$$X_i = e^{-Y_i/d} \quad (5.21)$$

for  $i = 1, \dots, N$ . Then we have

$$e^{-Y_P} = e^{t/d} X_1 X_2 \dots X_N \quad (5.22)$$

and therefore

$$W = X_1^d + X_2^d + \dots + X_N^d + e^{t/d} X_1 X_2 \dots X_N \quad (5.23)$$

However, we have to note that the field redefinition (5.21) is not single valued. In fact as we shift  $Y_i \rightarrow Y_i + 2\pi i$  we transform  $X_i$  by a primitive  $d$ -th root of unity

$$X_i \rightarrow X_i e^{-2\pi i/d}.$$

However, from (5.22) we see that the product of the  $X_i$  is well defined. Thus the LG theory we get is given by the above superpotential  $W$  modded out by  $G = (\mathbf{Z}_d)^{N-1}$  where  $G$  is the group which acts on  $X_i$  by all  $d$ -th roots of unity which preserve the product  $X_1 X_2 \dots X_N$ .

As we discussed in section 4, the linear sigma model for the above non-compact theory and the one given by a hypersurface of degree  $d$  in  $\mathbf{CP}^{N-1}$  differ by deformations involving only the  $cc$  ring elements, which thus do not affect the  $ac$  ring, which we are studying through mirror symmetry. Thus we expect the same LG with superpotential (5.23) (modded out by  $G$ ) to describe the mirror of hypersurface of degree  $d$  in  $\mathbf{CP}^{N-1}$ . Note that for the case of Calabi-Yau manifolds  $d = N$ , and this gives the mirror construction of Greene and Plesser [4]. However clearly there should be a difference between non-compact case and the compact theory. For example their conformal central charges (in  $N = 2$  units) differ by 2 in the Calabi-Yau case. What we will find is that indeed there is a difference and this is reflected in what is the appropriate field variable. In the compact case (i.e. the hypersurface case) we will find that the fundamental fields are  $X_i$ , whereas in the non-compact case the fundamental fields are  $Y_i$  which are related to  $X_i$  as defined in (5.21). This will be discussed after our discussion of the realization of D-branes in LG theories and their associated “BPS masses”.

$\mathcal{O}(-1) + \mathcal{O}(-1)$  over  $\mathbf{CP}^1$

As a special case of interest, which has been studied in [10] let us consider the mirror for  $\mathcal{O}(-1) + \mathcal{O}(-1)$  over  $\mathbf{CP}^1$ . This is given by the linear sigma model with a single  $U(1)$  with four fields with charges  $(-1, -1, +1, +1)$ . For non-vanishing  $t$  we can eliminate  $\Sigma$  as before and we obtain, in the dual formulation the superpotential

$$W = X_1 + X_2 + X_3 + X_4$$

with  $X_i = e^{-Y_i}$  and

$$X_1 X_2 = X_3 X_4 e^t.$$

We can eliminate, say  $X_1$  and write

$$W = X_2 + X_3 + X_4 + e^t X_3 X_4 / X_2$$

Later in this paper we will return to this example and show how the prepotential for this model (i.e., the famous tri-logarithmic structure of [11, 85]) can be obtained using the mirror potential we have found here.

## 6 D-branes, BPS Mass and LG Models

In the case of Calabi-Yau manifolds one can consider special  $n$ -dimensional Lagrangian submanifolds, and they represent a class in  $H_n(M)$ . We can imagine a D-brane wrapping that class. In the sigma model we consider worldsheet with boundaries where the boundary can end on these manifolds. Such boundaries preserve the A-model supercharges. Given the fact that the neutral observables of the  $B$  model correspond to  $n$ -form (after contraction with the holomorphic  $n$ -form) we have a natural pairing between  $B$ -model chiral fields, and  $A$ -model boundary states. The pairing is simply given by integrating the corresponding  $n$ -form on the corresponding  $n$ -cycle. Moreover, this integration can also be interpreted as the inner product of the boundary state defined by the D-brane and state defined (through topological twisting) by the  $B$ -model observable [86]. In other words

$$\langle \gamma_i | \phi_\alpha \rangle = \int_{\gamma_i} \phi_\alpha$$

where  $\gamma_i$  represents an  $n$ -dimensional cycle giving rise to boundary state  $\langle \gamma_i |$  and  $\phi_\alpha$  represents an element of the B-model chiral field corresponding to an  $n$ -form. There is one distinguished element for the chiral ring, the identity operator, which corresponds to the holomorphic  $n$ -form  $\Omega$  on the Calabi-Yau. In particular we have

$$Z_i = \langle \gamma_i | 1 \rangle = \int_{\gamma_i} \Omega$$



For type IIB compactifications on Calabi-Yau 3-folds, the  $Z_i$  also represents central extension of supersymmetry algebra, in a sector with D-brane wrapping the  $\gamma_i$  cycle.

In the context of LG models, the natural topological theory corresponds to the B-model one [56]. It is thus natural to ask the analog of this pairing in the context of the LG models. This is important in our applications because we have found an equivalence between the linear sigma models and LG models.

Consider a Landau Ginzburg theory with superpotential  $W(x_i)$  where  $x_i$  are chiral fields, with  $i = 1, \dots, n$ . Then it is known that the chiral ring for this theory is given by [7, 8]:

$$\mathcal{R} = \mathbf{C}[x_i]/dW$$

i.e., the ring generated by  $x_i$  subject to setting  $dW = 0$ . It is also known [87] that there is a corresponding homology group which is equal to the dimension of the ring, namely consider the homology group

$$H_n(\mathbf{C}^n, B)$$

where  $B$  denotes the asymptotic region in  $\mathbf{C}^n$  where  $ReW \rightarrow +\infty$ , i.e.  $n$ -cycles in the  $x_i$  space with boundaries ending on  $B$ . Then the dimension of this space is equal to the dimension of the ring  $\mathcal{R}$ . In fact as we will now discuss, there is a natural pairing between them and this identifies them as dual spaces. This is completely parallel to the case of the Calabi-Yau case and the pairing between mid-dimensional cycles and B-model observables (with equal left/right  $U(1)$  charge).

As is shown in [79], in fact we can construct analogs of Lagrangian submanifolds in the context of LG models as well, representing a basis for  $H_n(\mathbf{C}^n, B)$  and preserving the A-combination of supercharges  $Q_{ac} = Q_- + \overline{Q}_+$ . Moreover the image of these cycles on the  $W$ -plane is a straight line extending from the critical values of  $W$  to infinity along the positive real axis. Let  $\gamma_i$  represent one such cycle. We would like now to discuss the analog of the pairing discussed above for the case of Calabi-Yau manifolds, between the B-model observables and A-model boundary states.

Consider the pairing

$$\Pi_{i,a} = \langle \gamma_i | \phi_a \rangle$$

where  $\phi^a$  denotes a B-model chiral field observable and  $\langle \gamma_i |$  denotes the boundary state corresponding to  $\gamma_i$ . It has been shown in [57] that the  $\Pi_b$  satisfy the flatness equation in the context of  $tt^*$ :

$$\nabla_a \Pi_b = (D_a + C_a) \Pi_b = 0 \quad \overline{\nabla}_a \overline{\Pi}_b = (\overline{D}_a + \overline{C}_a) \overline{\Pi}_b = 0$$

where  $C_a$  denotes the multiplication by the chiral field corresponding to  $\phi_a$  on the chiral fields and  $D_a$  is a covariant derivative defined in [57]. This was formulated in [57] by showing that the above quantity can be computed by reducing to the case of quantum mechanics. The same result can also be derived more directly using the gymnastics leading to  $tt^*$  equation. In fact for the conformal case this has already been shown in [86] and the same argument also applies to the massive cases (by considering the geometry of semi-infinite cigar, as in the derivation of  $tt^*$ , with topological B-twisting on the tip of the cigar, as will be discussed in [79]).

A special case of this overlap, is given by  $\Pi_{i,1}$ , i.e. overlap with the state corresponding to the identity operator of the B-model. It has been shown in [57] that in the conformal case  $\Pi_{i,1}$  is a holomorphic function of the couplings, i.e. it is independent of the coefficients of the superpotential  $\overline{W}$  and depends only on the holomorphic couplings appearing in  $W$ . In the non-conformal case, this is no longer true. However even in the massive case one can obtain a holomorphic object by expanding near the conformal limit. This means formally considering  $\overline{W} \rightarrow \overline{\lambda} \overline{W}$  and taking the limit  $\overline{\lambda} \rightarrow 0$ . From this point on, when we refer to  $\Pi_{i,a}$  we have in mind this limit. It has been shown in [57] that in such a case  $\Pi_{i,1}$  has a simple integral expression:

$$\Pi_{i,1} = \int_{\gamma_i} \exp(-W) dx_1 \dots dx_n. \quad (6.1)$$

This is the Landau-Ginzburg analog of the BPS mass for the D-brane given by  $\gamma_i$ . Moreover, with a good choice of basis for chiral fields of the B-model (called the topological or flat coordinates) one has [57]

$$\partial_{t^a} \Pi_{i,1} = \Pi_{i,a} = \int_{\gamma_i} \phi_a \exp(-W) dx_1 \dots dx_n, \quad (6.2)$$

$$\partial_{t^a} \partial_{t^b} \Pi_{i,1} = \partial_{t^b} \Pi_{i,a} = C_{ab}^c \Pi_{i,c}, \quad (6.3)$$

where  $\phi_a$  denotes the chiral field corresponding to the deformation given by  $t^a$ . Note that this implies that the periods  $\Pi_{i,a}$  have the full information about the chiral ring structure constants.

Note that in the limit  $W = 0$  the above expression for the period is the usual integral of the holomorphic  $n$ -form (which for  $\mathbf{C}^n$  is  $\Omega = dx_1 \dots dx_n$ ) on a cycle. The appearance of  $e^{-W}$  reflects the fact that in the presence of superpotential the supercharges get modified and field which corresponds to D-brane boundary conditions in addition to the delta function forcing  $x_r$ 's to the subspace  $\gamma_i$  includes an additional factor  $e^{-W}$ , i.e.  $e^{-W} \delta(x - \gamma)$  is needed to be  $Q_{ac} = Q_- + \overline{Q}_+$  invariant.

## 6.1 Picard-Fuchs Equations for Periods

Applying the previous discussion to the mirror of the non-compact toric varieties we have discussed, we can compute the periods of the branes in the mirror LG theory. We have the periods being given as

$$\Pi = \int \prod_{i,b} dY_i \prod_b \delta(\sum_i Q_{ib} Y_i - t_b) \exp(-\sum_i e^{-Y_i}), \quad (6.4)$$

where we have suppressed the indices for cycles  $\gamma_i$  and the identity operator associated with  $\Pi$ . Consider instead

$$\Pi(\mu_j, t_b) = \int \prod_{i,b} dY_i \prod_b \delta(\sum_i Q_{ib} Y_i - t_b) \exp(-\sum_i \mu_i e^{-Y_i}), \quad (6.5)$$

which satisfies for each  $b$  the equations

$$[\prod_{Q_{ib}>0} (\frac{\partial}{\partial \mu_i})^{Q_{ib}}] \Pi(\mu_j, t_b) = e^{-t_b} [\prod_{Q_{ib}<0} (\frac{\partial}{\partial \mu_i})^{-Q_{ib}}] \Pi(\mu_j, t_b). \quad (6.6)$$

On the other hand by a shift in  $Y_i$  by  $\log \mu_i$  we can get rid of the  $\mu_i$  dependence above, except for a shift in the delta function constraint. In other words we have

$$\Pi(\mu_i, t_b) = \Pi_{i,1}(1, t_b - \log \prod_i \mu_i^{Q_{ib}}), \quad (6.7)$$

which is the original periods we are interested in computing. This in particular means that (6.6) can be rewritten as differential operators involving the  $t_b$  parameters. Together with the boundary conditions for large  $t_b$ 's these give an effective way of computing the periods.

$\mathcal{O}(-3)$  over  $\mathbf{CP}^2$

Just as an example of the previous equations, we can consider the equations satisfied by the periods of the mirror of sigma model on  $\mathcal{O}(-3)$  over  $\mathbf{CP}^2$ . This is given by a  $U(1)$  gauge theory with 4 matter fields with charges  $(-3, 1, 1, 1)$ . Using (6.6) we have

$$\frac{\partial}{\partial \mu_2} \frac{\partial}{\partial \mu_3} \frac{\partial}{\partial \mu_4} \Pi = e^{-t} \frac{\partial^3}{\partial \mu_1^3} \Pi. \quad (6.8)$$

Defining  $\theta = -d/dt$  and noting that  $\Pi$  depends on  $\mu_i$  in the combination  $t - \log[\mu_2 \mu_3 \mu_4 / \mu_1^3]$  we obtain

$$\theta^3 \Pi = e^{-t} (3\theta + 2)(3\theta + 1)(3\theta) \Pi. \quad (6.9)$$

## 6.2 Prepotential for $\mathcal{O}(-1) + \mathcal{O}(-1)$ bundle over $\mathbf{CP}^1$

We will now use the periods of the mirror to compute the prepotential for the non-compact Calabi-Yau threefold given by  $\mathcal{O}(-1) + \mathcal{O}(-1)$  bundle over  $\mathbf{CP}^1$ . The prepotential for this model is known to be given by trilogarithm function

$$F(t) = P_3(t) + \sum_{n>0} e^{-nt}/n^3 \quad (6.10)$$

where  $P_3(t)$  is a polynomial of degree 3 in  $t$  (some of whose coefficients are ambiguous). The physical meaning of  $F(t)$  as far as D-branes are concerned is as follows:  $\Pi_2 = t$  denotes the (complexified) volume of D2-brane wrapping  $\mathbf{CP}^1$ , whereas

$$\Pi_4 = \frac{\partial F}{\partial t}, \quad (6.11)$$

which is the dual period, corresponds to the complexified quantum volume of a non-compact D4 brane intersecting the  $\mathbf{CP}^1$  at one point. The ambiguity in the coefficients of  $P_3(t)$  reflects the infinity of the volume of this D4 brane.

From the discussion of the BPS masses of D-branes, and the mirror for this model, which we found in the form of the LG model

$$W = X_2 + X_3 + X_4 + e^t X_3 X_4 / X_2 \quad (6.12)$$

we are led to consider the appropriate periods given by integrals

$$\Pi = \int \frac{dX_2 dX_3 dX_4}{X_2 X_3 X_4} e^{-W} \quad (6.13)$$

where the measure is obtained by noting that the correct variables are the  $Y_i$ 's. It is more convenient to consider  $\partial\Pi/\partial t$  and integrating over  $X_4$ :

$$\frac{\partial\Pi}{\partial t} = e^t \int \frac{dX_2 dX_3 dX_4}{X_2^2} e^{-(X_2 + X_3 + X_4 + e^t X_3 X_4 / X_2)} = e^t \int \frac{dX_2 dX_3}{X_2^2} \delta(1 + e^t \frac{X_3}{X_2}) e^{(-X_2 - X_3)} \quad (6.14)$$

performing the integral over  $X_3$  we have

$$\frac{\partial\Pi}{\partial t} = \int \frac{dX_2}{X_2} e^{-X_2(1-e^{-t})} \quad (6.15)$$

There are two cycle one can consider here: Integrating around the origin of  $X_2$  we obtain 1. This corresponds to the D2 brane BPS mass, namely  $\partial\Pi_2/\partial t = 1$ . The other period corresponds to integrating from 0 to  $\infty$  on the  $X_2$  plane, which is the dual cycle corresponding to D4 brane. To do that consider taking another derivative of  $\Pi$ :

$$\frac{\partial^2\Pi}{\partial t^2} = \frac{\partial^3 F}{\partial t^3} = -e^{-t} \int dX_2 e^{-X_2(1-e^{-t})} = -\frac{e^{-t}}{(1-e^{-t})} \quad (6.16)$$

In agreement with the known result for  $F(t)$  (6.10).

### 6.3 Compact versus Non-Compact

As we discussed in section 4 compact complete intersections in toric varieties are closely related to the corresponding non-compact ones. Moreover all the quantities that are unaffected by the  $cc$  deformations in the compact theory, should be computable in the non-compact theory. This is true at the level of embedding of twisted chiral operators of the compact theory in the non-compact one and the example is the relation (4.11) by [65] presented in section 4. The same should be true about the overlap of states with B-type boundary condition with the ground state vacua, which according to [86] is independent of  $cc$  deformations.

From the above discussion of periods, one natural state to consider is the one corresponding to the identity operator in the context of the compact theory. As discussed in section 4 this corresponds to the state  $|1\rangle_{Compact} \leftrightarrow |\delta\rangle_{Non-Compact}$  in the non-compact theory. We now consider the pairing between B-type boundary states and the ground states (represented using A-model) in the sigma model. The B-type boundary states in the non-compact theory  $\langle\tilde{\gamma}_i|$  will flow to some boundary states in the compact theory  $\langle\gamma_i|$ . By the  $\epsilon$  independence we thus have

$$\langle\gamma_i|\dots\rangle_{Compact} = \langle\tilde{\gamma}_i|\dots|\delta_1\dots\delta_k\rangle_{Non-Compact} \quad (6.17)$$

In particular we learn that the BPS mass (which corresponds to the overlap of the state associated to identity operator with D-brane boundary state) for the compact theory is given by

$$\Pi_{i,1} = \langle\gamma_i|1\rangle_{Compact} = \langle\tilde{\gamma}_i|\delta_1\dots\delta_k\rangle_{Non-Compact} \quad (6.18)$$

Since we have found the mirror of non-compact theory in terms of an LG model, from our previous discussion it follows that

$$\Pi_{i,1} = \int_{\tilde{\gamma}_i} \delta_1\dots\delta_k \exp(-W) \quad (6.19)$$

Note that this result, for the case of linear sigma models corresponding to degree  $d$  hypersurfaces in toric varieties described by a single  $U(1)$  gauge theory can also be written as

$$\Pi_{Compact} = d \frac{\partial}{\partial t} \Pi_{Non-Compact} \quad (6.20)$$

This result is well known in the context of local mirror symmetry for Calabi-Yau manifolds [30–32] and should be viewed as a generalization of it. The equations (6.19) and its specialization (6.20) will be the fundamental results we need in completing our discussion for deriving the mirror of hypersurfaces (and complete intersections) in toric varieties.

Note that using the fact that the periods of the non-compact theory satisfy appropriate Picard-Fuchs equations, as derived in (6.6) we can use the above result (6.20) to write a Picard-Fuchs equation satisfied by the periods of the mirror for the compact case.

## 7 Mirror Symmetry for Complete Intersections

In this section we complete our derivation of the mirror theory corresponding to complete intersections in toric varieties. First we discuss the simple case of degree  $d$  hypersurfaces in  $\mathbf{CP}^{N-1}$  to illustrate how the idea works. Then we show how a subset of complete intersection sigma models have a similar mirror in the form of a simple Landau-Ginzburg orbifolds in flat space. We then show that the most general construction can also be described in terms of LG theory on non-compact Calabi-Yau manifolds.

### 7.1 Hypersurfaces in $\mathbf{CP}^{N-1}$

Consider a degree  $d$  hypersurface in  $\mathbf{CP}^{N-1}$ . As discussed before, all the ring structure and BPS masses can be computed in the associated non-compact theory. We have  $N + 1$  dual matter fields  $Y_i$  with  $i = 1, \dots, N$  and  $Y_P$  and one  $\Sigma$  field. From (6.20) we can compute the BPS masses for the compact theory in the form

$$\begin{aligned} \Pi &= d \frac{d}{dt} \int d\Sigma dY_P \prod_{i=1}^N dY_i \exp(-\widetilde{W}) \\ &= d \frac{d}{dt} \int dY_P \prod_{i=1}^N dY_i \delta\left(\sum_{i=1}^N Y_i - dY_P - t\right) \exp\left(-\sum_{i=1}^N e^{-Y_i} - e^{-Y_P}\right). \end{aligned} \quad (7.1)$$

Constraint  $\sum_i Y_i - dY_P = t$  can be solved, as discussed earlier, by

$$e^{-Y_i} = X_i^d, \quad (7.2)$$

$$e^{-Y_P} = e^{t/d} X_1 \cdots X_N. \quad (7.3)$$

The map from  $X_i$  to  $e^{-Y_i}$  and  $e^{-Y_P}$  is one-to-one up to the action of  $(\mathbf{Z}_d)^{N-1}$  on  $X_i$  defined by

$$X_i \mapsto \omega_i X_i, \quad \omega_i^d = 1, \quad \omega_1 \cdots \omega_N = 1. \quad (7.4)$$

Then, we obtain

$$\begin{aligned} \Pi &= d \frac{d}{dt} \int \prod_{i=1}^N \frac{dX_i}{X_i} \exp\left(-\sum_{i=1}^N X_i^d - e^{t/d} \prod_{i=1}^N X_i\right) \\ &= e^{t/d} \int \prod_{i=1}^N dX_i \exp\left(-\sum_{i=1}^N X_i^d - e^{t/d} \prod_{i=1}^N X_i\right). \end{aligned} \quad (7.5)$$

This is the period for Landau-Ginzburg model with superpotential

$$W_{LG} = X_1^d + \cdots + X_N^d + e^{t/d} X_1 \cdots X_N, \quad (7.6)$$

or more precisely the LG orbifold by the  $(\mathbf{Z}_d)^{N-1}$  action (7.4). What we see here is that the compact theory and non-compact theory both have the same Landau-Ginzburg potential, but the measure for the fundamental fields, which determine the measure are different in the two cases. In other words the LG orbifold with fields  $X_i$  as the fundamental fields (rather than  $Y_i$  as in the non-compact case) has the same ring structure and BPS masses as the degree  $d$  hypersurfaces in  $\mathbf{CP}^{N-1}$ .

Let us examine the vacua of this LG orbifold. The equation  $dW_{LG} = 0$  has  $(N - d)$  solutions at

$$X_1^d = \cdots = X_N^d = -\frac{e^{t/d}}{d} X_1 \cdots X_N =: S, \quad S^{N-d} = (-d)^d e^{-t} \quad (7.7)$$

and, for  $d \geq 2$ , one solution at

$$X_1 = \cdots = X_N = 0. \quad (7.8)$$

The  $(N - d)$  critical points (7.7) are all non-degenerate and correspond to massive vacua that break spontaneously the  $\mathbf{Z}_{2(N-d)}$  axial R-symmetry to  $\mathbf{Z}_2$ . The critical point at  $X_i = 0$  is degenerate for  $d > 2$  and corresponds to a non-trivial fixed point. The behaviour of the theory in the IR limit depends on the relation of  $d$  and  $N$ , as follows.

- For  $d = 1$  where  $M = \mathbf{CP}^{N-2}$ , there are only  $N - 1$  massive vacua. If we integrate out  $X_N$ , we obtain the constraint  $X_{N-1} = -e^{-t}/(X_1 \cdots X_{N-2})$  and the effective superpotential for the remaining fields is

$$W = X_1 + \cdots + X_{N-2} - e^{-t}/(X_1 \cdots X_{N-2}). \quad (7.9)$$

This is nothing but the  $A_{N-2}$  affine Toda superpotential. Thus, we have reproduced the mirror symmetry of  $\mathbf{CP}^{N-2}$  model and affine Toda theory.

- For  $2 \leq d < N$  where  $M$  has positive first Chern class and the sigma model is asymptotically free, there are  $(N - d)$  massive vacua and a vacuum at  $X_i = 0$ . At  $X_i = 0$ , the last term  $e^{t/d} X_1 \cdots X_N$  of the potential (7.6) is irrelevant compared to the first  $N$ -terms. This can be seen by computing its dimension in the superconformal field theory we will study below. (More intuitively, this is because  $t$  flows at low energies to large negative values.) Thus, the theory at  $X_i = 0$  corresponds to the  $(\mathbf{Z}_d)^{N-1}$ -orbifold of the LG model with the quasi-homogeneous superpotential

$$W_{IR} = X_1^d + \cdots + X_N^d. \quad (7.10)$$

In particular, we see the enhancement of the axial  $\mathbf{Z}_{2(N-d)}$  R-symmetry to  $U(1)_A$  symmetry. For  $d = 2$  the critical point at  $X_i = 0$  is non-degenerate and does not correspond to a non-trivial fixed point. However, because of the orbifolding, it could correspond to multiple vacua.

- For  $d = N$  where  $M$  is a Calabi-Yau manifold, there are no massive vacua but one massless vacuum at  $X_i = 0$ . At  $X_i = 0$ , the last term of (7.6) is equally relevant compared to the first  $N$  terms. Thus  $t$  remains as the parameter describing a marginal deformation of the SCFT.
- For  $d > N$  where  $M$  has negative first Chern class, there are  $(d - N)$  massive vacua and one massless vacuum at  $X_i = 0$ . At  $X_i = 0$ , the first  $N$  terms of (7.6) are irrelevant compared to the last term. Thus, the IR fixed point is described by the theory with superpotential

$$W_{IR} = e^{t/d} X_1 \cdots X_N. \quad (7.11)$$

The vacuum equation  $dW_{IR} = 0$  is solved if two of  $X_i$ 's vanish. Namely, the theory is a free SCFT on  $\mathbf{C}^{N-2}$ . This is expected since if  $c_1(M) < 0$  the sigma model is IR free.

Thus, we have seen that the LG orbifold with the superpotential (7.6) and group  $(\mathbf{Z}_d)^{N-1}$  captures the physics of the sigma model for all values of  $d$ . Also, it describes *both* the massive vacua *and* the massless vacuum that flows to a non-trivial (or trivial) IR fixed point. Below, we shall analyze the spectrum of the SCFT with the superpotential (7.10) that appears as the non-trivial fixed point for hypersurfaces with  $c_1 > 0$ .

### 7.1.1 The Spectrum of Chiral Primary Fields for Hypersurfaces in $\mathbf{CP}^{N-1}$

We can use the method developed in [44, 88] to analyze the spectrum of the chiral primary fields. We are considering the LG model with the superpotential

$$W = X_1^d + \cdots + X_N^d \quad (7.12)$$

divided by the orbifold group  $(\mathbf{Z}_d)^{N-1}$  acting on  $X_i$ 's as  $X_i \mapsto \omega^{\alpha_i} X_i$  where

$$\omega = e^{2\pi i/d}, \quad \sum_{i=1}^N \alpha_i = 0 \pmod{d}. \quad (7.13)$$

Recall that we are in the non-standard convention of chirality as the LG model; the fields  $X_i$  are *twisted chiral* superfields of R-charge  $-q_i = \bar{q}_i = 1/d$ .<sup>1</sup> The model before

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<sup>1</sup> $q$  and  $\bar{q}$  are the left/right R-charges related to the vector/axial R-charges by  $q_V = \bar{q} + q$  and  $q_A = \bar{q} - q$ .



orbifolding has only the *ac* ring given by

$$\mathcal{R} = \mathbf{C}[X_1, \dots, X_N] / \partial_i W = \bigoplus_{0 \leq p_i \leq d-2} \mathbf{C} X_1^{p_1} \cdots X_N^{p_N}, \quad (7.14)$$

and the central charge is  $c/3 = \sum_i (1 - 2/d) = N(d - 2)/d$ .

### *Supersymmetric Ground States*

The supersymmetric (or Ramond) ground states of the  $(\mathbf{Z}_d)^{N-1}$  orbifold come both from untwisted and twisted sectors.

The Ramond ground states in the untwisted sector are build from the canonical ground state  $|0\rangle_R^1$  of R-charge  $\bar{q} = -q = -c/6 = -N(d - 2)/2d$  by multiplying the  $(\mathbf{Z}_d)^{N-1}$  invariant elements of (7.14). Obviously they are the powers of  $X_1 \cdots X_N$ . Thus, there are  $d - 1$  Ramond ground states in the untwisted sector;

$$(X_1 \cdots X_N)^p |0\rangle_R^1, \quad p = 0, 1, \dots, d - 2. \quad (7.15)$$

The R-charge is just the sum of  $-c/6$  and  $Np(1/d)$ ;

$$\bar{q}_p = -q_p = \frac{N}{d} \left( p - \frac{d-2}{2} \right). \quad (7.16)$$

Let us next consider the  $h$ -twisted sector for  $h = (\omega^{\alpha_1}, \dots, \omega^{\alpha_N}) \in (\mathbf{Z}_d)^{N-1}$ . The  $h$ -invariant fields are  $X_i$  with  $\omega^{\alpha_i} = 1$ . There is a (not necessarily physical) ground state  $|0\rangle_R^h$  in the sector on which  $g \in (\mathbf{Z}_d)^{N-1}$  acts as the multiplication by  $\det g|_h$  where  $g|_h$  is the action of  $g$  restricted on  $h$ -invariant fields. ( $\det g|_h = 1$  if there is no  $h$ -invariant field). The Ramond ground states in the  $h$ -twisted sector are  $(\mathbf{Z}_d)^{N-1}$  invariant states of the form  $X_{i_1}^{p_1} \cdots X_{i_s}^{p_s} |0\rangle_R^h$  where  $X_{i_a}$ 's are  $h$ -invariant fields and  $0 \leq p_a \leq d - 2$ . If  $\omega^{\alpha_i} = 1$  for some  $i$  (say  $i = 1$ ), then  $\omega^{\alpha_j} \neq 1$  for some other  $j$  (say  $j = N$ ) if  $h \neq 1$ . Then  $X_1$  is  $h$ -invariant but  $X_N$  is not. For  $g = (\omega, 1, \dots, 1, \omega^{-1})$ , we have  $\det g|_h = \omega$ . But then  $g$  acts on the state  $X_1^{p_1} \cdots |0\rangle_R^h$  by multiplication by  $\omega^{p_1+1} \neq 1$ . So there is no  $(\mathbf{Z}_d)^{N-1}$ -invariant state. Thus, Ramond ground states exist only for  $h$  such that  $\omega^{\alpha_i} \neq 1$  for all  $i$ . For such an  $h$ , there is no  $h$ -invariant field and therefore there is a unique Ramond ground state

$$|0\rangle_R^h. \quad (7.17)$$

Its R-charge can be obtained from the formula (3.2) in [88]

$$\bar{q}_h = q_h = - \left( \frac{\sum_i \alpha_i}{d} - \frac{N}{2} \right). \quad (7.18)$$

Since  $\sum_i \alpha_i = 0 \bmod d$ , we have  $\sum_i \alpha_i = n_h d$  where  $n_h$  is an integer in the range  $N/d \leq n_h \leq N(1 - 1/d)$ . Thus, the vector R-charge  $q^V = \bar{q}_h + q_h$  is an integer  $N - 2n_h$  in the range

$$|q_h^V| \leq c/3 < N - 2. \quad (7.19)$$

Let us count the number of such  $h \in (\mathbf{Z}_d)^{N-1}$ . Let  $\mathcal{C}_N$  be the set of such  $h$ 's. Let us consider a sequence  $(\omega^{\alpha_1}, \dots, \omega^{\alpha_{N-1}})$  such that any entry is not equal to 1. If  $\alpha_* := \sum \alpha_i$  is equal to 0 mod  $d$ , this determines an element of  $\mathcal{C}_{N-1}$ . Otherwise this determines a unique element  $(\omega^{\alpha_1}, \dots, \omega^{\alpha_{N-1}}, \omega^{-\alpha_*})$  of  $\mathcal{C}_N$ . Thus we obtain the recursion relation  $(d-1)^{N-1} = \#(\mathcal{C}_{N-1}) + \#(\mathcal{C}_N)$  which is solved by

$$\#(\mathcal{C}_N) = (-1)^N \frac{(1-d)^N - (1-d)}{d}. \quad (7.20)$$

This is the total number of Ramond ground states from twisted sectors.

Let us compute the Witten index  $\text{Tr}(-1)^F$ . We normalize  $(-1)^F = 1$  on  $|0\rangle_R^1$ . Then the ground states from untwisted sector has  $(-1)^F = (-1)^N$  [88]. Thus, the index of the SCFT is

$$\text{Tr}_{\text{SCFT}}(-1)^F = (d-1) + \frac{(1-d)^N - (1-d)}{d} = \frac{(1-d)^N + d^2 - 1}{d}. \quad (7.21)$$

Together with the  $(N-d)$  massive vacua having  $(-1)^F = 1$ , we obtain the total index

$$\text{Tr}(-1)^F = (N-d) + \frac{(1-d)^N + d^2 - 1}{d} = \frac{(1-d)^N + Nd - 1}{d}. \quad (7.22)$$

This indeed agrees with the Euler number of the hypersurface  $M$ .

Note that our original sigma model possesses the unbroken  $U(1)_V$  R-symmetry and it counts  $-p+q$  of the Hodge number  $(p, q)$  of the harmonic form representing a vacuum. For a hypersurface in  $\mathbf{CP}^{N-1}$  the off diagonal Hodge number is non-zero only for  $p+q = N-2$ . Thus, the ground state  $|0\rangle_R^h$  with non-zero  $q_h^V = N - 2n_h$  corresponds to a harmonic  $(p, q)$ -form with  $q-p = q_h^V$  and  $p+q = N-2$ . In other words, the number of such  $h$ 's for a fixed  $(p, q)$  must be equal to the Hodge number  $h^{p,q}$ . This is indeed easy to check explicitly for small values of  $N$  and  $d$  (e.g. using a formula for the Hodge numbers in [89]). For illustration, let us present the result for two cases  $N = 4, d = 3$  and  $N = 5, d = 3$ :

$N=4, d=3$

$M$  is the cubic surface in  $\mathbf{CP}^3$  which is known as  $E_6$  del Pezzo surface. It has  $h^{0,0} = h^{2,2} = 1, h^{1,1} = 7$  and the off-diagonal Hodge numbers are all zero. Thus, classically there are one ground state with  $(-q, \bar{q}) = (\pm 1, \pm 1)$  and seven ground states with  $(-q, \bar{q}) = (0, 0)$ . In the quantum theory there is a single massive vacuum and a massless vacuum.

The massless vacuum corresponds to a SCFT of  $c/3 = 4/3$ . There are one untwisted ground state with  $(-q, \bar{q}) = (\pm 2/3, \pm 2/3)$  and six twisted ground states with  $(-q, \bar{q}) = (0, 0)$ .

$N=5, d=3$

$M$  is a cubic hypersurface in  $\mathbf{CP}^4$  which has  $h^{p,p} = 1$  for  $p = 0, 1, 2, 3$  and  $h^{2,1} = h^{1,2} = 5$ . Thus, classically there are one ground state with  $(-q, \bar{q}) = (p - 3/2, p - 3/2)$  for  $p = 0, 1, 2, 3$ , five with  $(-q, \bar{q}) = (1/2, -1/2)$  and five with  $(-q, \bar{q}) = (-1/2, 1/2)$ . In the quantum theory, there are two massive vacua and a massless vacuum. The massless vacuum corresponds to a SCFT with  $c/3 = 5/3$ . There are one untwisted ground state with  $(-q, \bar{q}) = (\pm 5/6, \pm 5/6)$ , five twisted states with  $(-q, \bar{q}) = (1/2, -1/2)$  and five twisted states with  $(-q, \bar{q}) = (-1/2, 1/2)$ .

*ac Primaries*

Since all the vector R-charges are integral, the *ac* primary states are in one to one correspondence with the Ramond ground states by spectral flow. The spectral flow simply changes the R-charges by  $\bar{q}_{ac} = \bar{q}_R + c/6$  and  $q_{ac} = q_R - c/6$ . Thus, we have the following *ac* primary states with R-charges: From the untwisted sector

$$(X_1 \cdots X_N)^p |0\rangle_{ac}^1; \quad \bar{q}_p = -q_p = \frac{Np}{d}, \quad (7.23)$$

( $p = 0, \dots, d-2$ ), and from the twisted sectors

$$|0\rangle_{ac}^h; \quad \bar{q}_h = q_h + c/3 = N \left(1 - \frac{1}{d}\right) - n_h. \quad (7.24)$$

where  $n_h$  is the integer in the range  $N/d \leq n_h \leq N(1 - 1/d)$  defined above.

*cc Primaries*

Since the axial R-charges are not necessarily integers, we must separately consider *cc* primary states. The *cc* primaries in the  $h$ -twisted sector for  $h = (\omega^{\alpha_1}, \dots, \omega^{\alpha_N})$  (including  $h = 1$ ) can be found in the same way as the search for Ramond ground states in  $hj^{-1}$ -twisted sector where  $j = (\omega, \dots, \omega)$  [88]. There is a unique *cc* primary

$$|0\rangle_{cc}^h, \quad (7.25)$$

for each  $h$  such that  $\beta_i := \alpha_i - 1$  obey  $\sum_{i=1}^N \beta_i = -N$  and  $\beta_i \not\equiv 0 \pmod{d}$ . One can choose  $\beta_i$  in the range  $1 \leq \beta_i \leq d-1$  and then  $\sum_i \beta_i = -N + m_h d$  where  $m_h$  is an integer in the

range  $2N/d \leq m_h \leq N$ . The R-charge of the  $cc$  primary state is then

$$\bar{q} = q = \frac{c}{6} - \sum_{i=1}^N \left( \frac{\beta_i}{d} - \frac{1}{2} \right) = N - m_h. \quad (7.26)$$

If  $N$  is divisible by  $d$ , there are extra  $cc$  primaries from the  $h = (\omega, \dots, \omega)$ -twisted sector;

$$(X_1 \cdots X_N)^p |0\rangle_{cc}^h, \quad p = 0, \dots, d-2. \quad (7.27)$$

which have R-charges

$$\begin{pmatrix} \bar{q} \\ q \end{pmatrix} = \frac{c}{12} \mp \frac{c}{12} \pm \frac{Np}{d}. \quad (7.28)$$

The R-charges (7.26) and (7.28) are integers in the range  $0 \leq q, \bar{q} \leq c/6$ .

### *Marginal Deformations*

The marginal deformation of the theory preserving the  $(2, 2)$  superconformal symmetry is done by an  $ac$  primary field of  $\bar{q} = -q = 1$  or a  $cc$  primary field of  $\bar{q} = q = 1$ .

From the above list, it is easy to see that there are no such  $ac$  primaries except the special case  $N = 6, d = 3$  where twenty  $n_h = 3$  twisted states does correspond to the marginal deformation. This is the case which corresponds to the conformal field theory of  $K3$  [90].

On the other hand, there are always  $cc$  primaries with  $\bar{q} = q = 1$ ; they are the states (7.25) with  $m_h = N - 1$ . This corresponds to  $\alpha_i$ 's such that  $\sum_i (d - \alpha_i) = d$  for  $0 \leq d - \alpha_i \leq d - 2$ . The number of such  $\alpha_i$ 's is the same as the number of independent polynomial deformations of degree  $d$  equation in  $N$  variables. As we will see below, they correspond to the complex structure deformations of the hypersurface  $M$ .

Incidentally all the cases where  $d$  divides  $N$  gives rise to a conformal field theory in the IR which is mirror to a Calabi-Yau manifold. Let  $N/d = k$  be an integer. Then we obtain a conformal field theory corresponding to a Calabi-Yau with complex dimension  $(d-2)k$ . In fact it is mirror to a Calabi-Yau which is an orbifold of  $k$  copies of degree  $d$  hypersurface in  $\mathbf{CP}^{d-1}$ . For example, as noted in [90] the case  $N = 9, d = 3$  is the mirror to a rigid Calabi-Yau threefold. For such cases we can now find a geometric interpretation of the mirror: *The mirror of these rigid Calabi-Yau manifolds can be viewed as the infrared limit of sigma model on the corresponding hypersurface with positive  $c_1$ .*

### 7.1.2 Mirror Symmetry of Orbifold Minimal Models as IR Duality

An LG description of the IR fixed point in this model is actually available also in the original linear sigma model [43] (see also [91]). This is the extension of the basic argument for CY/LG correspondence [10] to the sigma model of  $c_1 > 0$  hypersurfaces in  $\mathbf{CP}^{N-1}$ .

For  $d < N$ , the Kahler parameter  $r$  flows at low energies to large negative values  $r \ll 0$ . There we find  $(N - d)$  massive vacua at large values of  $\sigma$ ;  $\sigma^{N-d} = e^{-t}(-d)^d$ . Actually, there is also a vacuum at  $\sigma = 0$  but  $|p|^2 = |r|/d$ . (Presumably the equation for  $\sigma$  should be replaced by  $\sigma^{N-1} = e^{-t}(-d)^d \sigma^{d-1}$ , which is the chiral ring relation for the sigma model on the hypersurface  $M$  [92]. This indeed has a solution at  $\sigma = 0$  for  $d > 1$ .) Since  $p$  has electric charge  $-d$ , the  $U(1)$  gauge symmetry is broken to  $\mathbf{Z}_d$  by the Higgs mechanism. The superfield  $P$  is massive while other fields of unit charge  $\Phi_i$  are massless and become the relevant fields to describe the low energy theory. Because of the expectation value of  $P$ , there is a non-trivial superpotential for  $\Phi_i$ 's

$$W = \sqrt{|r|/d} (\Phi_1^d + \cdots + \Phi_N^d). \quad (7.29)$$

Namely, the theory flows in the IR limit to  $(N - d)$  empty theories and the  $\mathbf{Z}_d$  orbifold of the LG model with the superpotential (7.29). This is very similar to the structure found in our effective theory (7.6) except that the group is now a single  $\mathbf{Z}_d$ . Since the two LG orbifolds, one by  $\mathbf{Z}_d$  and the other by  $(\mathbf{Z}_d)^{N-1}$ , arise as IR fixed points of the same theory, they must agree with each other. Namely, the  $\mathbf{Z}_d$ -orbifold and the  $(\mathbf{Z}_d)^{N-1}$ -orbifold must be mirror to each other.<sup>2</sup> This mirror symmetry is actually the special case of the mirror symmetry between orbifolds of minimal models mentioned in section 2.

It is straightforward to compute the spectrum of chiral primary fields in the  $\mathbf{Z}_d$  orbifold model. The result is of course in agreement with the one for the  $(\mathbf{Z}_d)^{N-1}$  orbifold. Untwisted sector states in the  $\mathbf{Z}_d$  orbifold corresponds to twisted sector states in the  $(\mathbf{Z}_d)^{N-1}$  orbifold, and vice versa. For instance, the untwisted  $cc$  primary field  $\Phi_1^{p_1} \cdots \Phi_N^{p_N}$  ( $0 \leq p_i \leq d - 2$ ) corresponds to the  $h = (\omega^{\alpha_1}, \dots, \omega^{\alpha_N})$ -twisted  $cc$  primary state (7.25) in the  $(\mathbf{Z}_d)^{N-1}$  orbifold where  $p_i = d - \alpha_i$ . This is actually what is expected since  $\Phi_i$  and  $e^{-Y_i} = X_i^d$  are dual to each other and a momentum (power) of one corresponds to a winding (twist) of the other.

The interpretation of the marginal deformation by  $cc$  primaries is clear in this picture. The marginal  $cc$  deformation corresponds to deformation of the LG model (7.29) by  $\Phi_1^{p_1} \cdots \Phi_N^{p_N}$  with  $\sum_i p_i = d$  and  $0 \leq p_i \leq d - 2$ . This corresponds to the deformation of the superpotential by  $P\Phi_1^{p_1} \cdots \Phi_N^{p_N}$  in the original linear sigma model for  $M$  which is nothing

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<sup>2</sup> $\Phi_i$  are *chiral* superfields and the model (7.29) has the standard convention of chirality.

but the polynomial deformation of the defining equation of  $M$ . Actually, any deformation of the complex structure is of this form.<sup>3</sup> Thus, the  $cc$  deformation in the sigma model on a  $c_1 > 0$  projective hypersurface is in one to one correspondence with the complex structure deformation, as in the case of Calabi-Yau sigma models. It would be interesting to better understand why this holds in this case and investigate whether this is a general fact.

## 7.2 Complete Intersections

LG orbifold description of the mirror is possible also for a class of complete intersections in toric varieties. Let  $X$  be a toric variety defined by the charge matrix  $Q_{ia}$  ( $i = 1, \dots, N$ ,  $a = 1, \dots, k$ ). We consider a complete intersection  $M$  in  $X$  defined by the equations  $G_b = 0$  ( $b = 1, \dots, k$ ) as many as the number of the  $U(1)$  gauge groups to define  $X$ . Here  $G_b$  is a polynomial of  $\Phi_i$  of a certain “degree”  $d_{ba}$  (that is,  $G_b$  has charge  $d_{ba}$  for the  $a$ -th  $U(1)$  gauge group) where we assume  $d_{ba}$  to be an invertible matrix.

The dual of the non-compact theory is described by  $N + k$  twisted chiral fields  $Y_i$  and  $Y_{P_b}$  dual to  $\Phi_i$  and  $P_b$  and  $k$  field strengths  $\Sigma_a$ . It has the twisted superpotential

$$\widetilde{W} = \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^N Q_{ia} Y_i - \sum_{b=1}^k d_{ba} Y_{P_b} - t_a \right) + \sum_{i=1}^N e^{-Y_i} + \sum_{b=1}^k e^{-Y_{P_b}}. \quad (7.30)$$

The period integral relevant for the compact theory is given by

$$\Pi = \int \prod_{a=1}^k d\Sigma_a \prod_{i=1}^N dY_i \prod_{b=1}^k dY_{P_b} \delta_1 \cdots \delta_k \exp(-\widetilde{W}), \quad (7.31)$$

where  $\delta_b = \sum_{a=1}^k d_{ba} \Sigma_a$ . Then, the period is expressed as follows (a quick derivation of this is given in the next subsection in more general cases);

$$\begin{aligned} \Pi &= \int \prod_{i=1}^N dY_i \prod_{b=1}^k dY_{P_b} e^{-Y_{P_1}} \cdots e^{-Y_{P_k}} \\ &\quad \times \prod_{a=1}^k \delta \left( \sum_{i=1}^N Q_{ia} Y_i - \sum_{b=1}^k d_{ba} Y_{P_b} - t_a \right) \times \exp \left( - \sum_{i=1}^N e^{-Y_i} - \sum_{b=1}^k e^{-Y_{P_b}} \right). \end{aligned} \quad (7.32)$$

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<sup>3</sup>This can be seen as follows (we thank R. Pandharipande for explanation). Using the long exact sequence for  $0 \rightarrow T_M \rightarrow T_{\mathbb{CP}^{N-1}|M} \rightarrow \mathcal{O}_M(d) \rightarrow 0$ , it is enough to show  $H^1(M, T_{\mathbb{CP}^{N-1}|M}) = 0$ . From  $0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus N} \rightarrow T_{\mathbb{CP}^{N-1}} \rightarrow 0$ , this reduces to  $H^1(M, \mathcal{O}_M(1)) = 0$  and  $H^2(M, \mathcal{O}_M) = 0$ . These further reduce via  $0 \rightarrow \mathcal{O}(-d) \rightarrow \mathcal{O} \rightarrow \mathcal{O}_M \rightarrow 0$  to  $H^{1+i}(\mathbb{CP}^{N-1}, \mathcal{O}(1-i)) = 0$  and  $H^{2+i}(\mathbb{CP}^{N-1}, \mathcal{O}(1-i)) = 0$  for  $i = 0, 1$ . That this holds for  $d < N$  is a standard fact (e.g. by Kodaira-Nakano vanishing theorem [93]). Within  $d \leq N$ , the only case this fails is  $N = d = 4$ , the famous example  $M = K3$ .

Suppose there are matrices  $m_{ji}$ ,  $n_{jb}$  of non-negative integers such that

$$\sum_{i=1}^N m_{ji} Q_{ia} = \sum_{b=1}^k n_{jb} d_{ba}. \quad (7.33)$$

Then, the constraints for  $Y_i$  and  $Y_{P_b}$  can be solved as

$$e^{-Y_i} = \prod_{j=1}^N X_j^{m_{ji}}, \quad e^{-Y_{P_b}} = e^{d_{ab}^{-1} t_a} \prod_{j=1}^N X_j^{n_{jb}}. \quad (7.34)$$

Furthermore, if  $m_{ji}$  is an invertible matrix and  $\sum_b n_{jb} = 1$  for each  $j$  (thus  $n_{jb}$  is either 0 or 1), the period integral can be expressed as

$$\Pi = \int \prod_{i=1}^N dX_i \exp(-\widetilde{W}), \quad (7.35)$$

for

$$\widetilde{W} = \sum_{i=1}^N \prod_{j=1}^N X_j^{m_{ji}} + \sum_{b=1}^k e^{d_{ab}^{-1} t_a} \prod_{j=1}^N X_j^{n_{jb}}. \quad (7.36)$$

Thus, under the condition that the charges  $Q_{ia}$  and the degrees  $d_{ba}$  admit such matrices  $m_{ji}$  and  $n_{jb}$  as the solution to (7.33), the period of the dual theory is the same as that of the LG orbifold with the twisted chiral superpotential (7.36). The orbifold group  $\tilde{\Gamma}$  is generated by  $X_j \rightarrow \omega_j X_j$  where  $\omega_j$  are phases satisfying  $\prod_j \omega_j^{m_{ji}} = 1$  and  $\prod_j \omega_j^{n_{jb}} = 1$ .

As in the example of hypersurfaces in  $\mathbf{CP}^{N-1}$ , one might expect that the last  $k$  terms in (7.36) are irrelevant at low energies and the IR fixed point is described by

$$\widetilde{W}_{IR} = \sum_{i=1}^N \prod_{j=1}^N X_j^{m_{ji}}. \quad (7.37)$$

However, as we will see, an extra condition is required for this. Intuitively, the last  $k$  terms are irrelevant when  $d_{ab}^{-1} t_a \rightarrow -\infty$  at low energies. The flow of  $t_a$  is determined by  $\beta_a = \sum_{i=1}^N Q_{ia} - \sum_{b=1}^k d_{ba}$  as  $t_a(\mu) = \beta_a \log(\mu/\Lambda)$ . Then, the condition of irrelevance is  $d_{ab}^{-1} \beta_a > 0$  or equivalently

$$\sum_{i,a} d_{ab}^{-1} Q_{ia} > 1. \quad (7.38)$$

More precisely, let  $2q_i$  be the axial charge of  $X_i$  so that each of the first  $N$  terms in (7.36) has charge 2, namely  $\sum_{j=1}^N m_{ji} q_j = 1$ . Then, from (7.33) we find  $\sum_{i=1}^N Q_{ia} = \sum_{b,j} q_j n_{jb} d_{ba}$ . Thus the condition (7.38) means  $\sum_{j=1}^N q_j n_{jb} > 1$ , which is nothing but the irrelevance of  $\prod_{j=1}^N X_j^{n_{jb}}$ . Thus, under the condition (7.38), the IR fixed point is described by the LG orbifold with superpotential (7.37). The central charge of this model is  $c/3 = N - 2 \sum_{i,j} m_{ij}^{-1}$ . What we have said remains true when the inequalities are relaxed

to allow equalities. If an equality holds, the corresponding term in (7.36) is a marginal operator and should be kept.

In the original linear sigma model, on the other hand, under the same condition we can take  $G_b$  to be

$$G_b = \sum_{j=1}^N n_{jb} \prod_{i=1}^N \Phi_i^{m_{ji}}, \quad (7.39)$$

where the sum is over  $j$  such that  $n_{jb} = 1$ . The D-term equations for the chiral fields are expressed as

$$\sum_{i,a} d_{ab}^{-1} Q_{ia} |\phi_i|^2 - |p_b|^2 = \sum_a d_{ab}^{-1} t_a. \quad (7.40)$$

At low energies, under the condition (7.38), the right hand side flows to large negative values for all  $b$ . Under the same condition, the coefficients of  $|\phi_i|^2$  are all positive. Then, the equation (7.40) implies all  $p_b$  are non-vanishing. The vacuum equation also requires  $\sum_{b=1}^k p_b \partial_i G_b = 0$ . It follows from this that  $\prod_i \Phi_i^{m_{ji}} = 0$  for all  $j$  and that  $\sum'_{j_1,b} n_{j_1 b} p_b \prod_{j_2 \neq i} \Phi_{j_2}^{m_{j_1 j_2}} = 0$  for all  $i$  where the sum  $\sum'$  is over such  $(j_1, b)$  that  $m_{j_1 i} = 1$ . We assume that this implies  $\Phi_i = 0$  for all  $i$ . (We do not attempt to prove it here. It is possible that in general an extra condition is required.) Then the gauge group  $U(1)^k$  is broken to its subgroup  $\Gamma$  (generated by  $g_a$  such that  $\prod_a g_a^{d_{ba}} = 1, \forall b$ ) which is a discrete subgroup since  $d_{ba}$  is invertible, and the massless degrees of freedom are  $\Phi_i$ 's only. Thus, we expect that the theory flows in the IR limit to the  $\Gamma$ -orbifold of the LG model with the superpotential

$$W = \sum_{j,b} n_{jb} \langle p_b \rangle \prod_{i=1}^N \Phi_i^{m_{ji}} \quad (7.41)$$

where  $\langle p_b \rangle$  is the expectation value of the massive field  $p_b$ . One can relax the inequality in (7.38) to admit equality. If an equality holds, we can *choose* the value of  $t_a$ 's such that  $d_{ab}^{-1} t_a$  are all negative, and the same conclusion holds. The central charge of the model is  $c/3 = N - 2 \sum_{i,j} m_{ij}^{-1}$ , the same as the one for (7.37).

Since the two LG orbifold models appear as the IR limit of the same theory, they must be equivalent, or mirror to each other. The equivalences of this type between LG models have been noted before [94, 95].

For illustration, let us present an example. We consider a complete intersection in  $\mathbf{CP}^{N-1} \times \mathbf{CP}^{M-1}$  ( $N \geq M$ ) defined by two equations of bi-degree  $(d_1, 0)$  and  $(1, d_2)$  respectively. It has a non-negative first Chern class if  $N \geq d_1 + 1$  and  $M \geq d_2$ . The equations for the homogeneous coordinates  $S_i$  ( $i = 1, \dots, N$ ) and  $T_j$  ( $j = 1, \dots, M$ ) are

$$G_1 = \sum_{i=1}^N S_i^{d_1}, \quad G_2 = \sum_{j=1}^M S_j T_j^{d_2}. \quad (7.42)$$



Under the condition<sup>1</sup>

$$N \geq d_1 + M/d_2, \quad M \geq d_2, \quad (7.43)$$

$d_{ab}^{-1}t_a$  are (or can be chosen) large negative at low energies, and we find  $p_1$  and  $p_2$  are non-vanishing. Then, we can show in this case that  $\sum_{b=1,2} p_b \partial G_b = 0$  implies that all  $S_i$  and  $T_j$  must be vanishing. Thus, we find an LG orbifold description of the low energy theory where the superpotential is

$$W = \sum_{j=1}^M (S_j^{d_1} + S_j T_j^{d_2}) + \sum_{k=M+1}^N S_k^{d_1}, \quad (7.44)$$

and the orbifold group is the subgroup of  $U(1)_1 \times U(1)_2$  defined by  $g_1^{d_1} = 1$  and  $g_1 g_2^{d_2} = 1$ . On the other hand, the dual theory is described by the LG orbifold for  $N + M$  twisted chiral superfields  $U_i$  and  $V_j$  with the twisted superpotential

$$\widetilde{W} = \sum_{j=1}^M (U_j^{d_1} V_j + V_j^{d_2}) + \sum_{k=M+1}^N U_k^{d_1} + e^{\frac{t_1}{d_1} - \frac{t_2}{d_1 d_2}} \prod_{i=1}^N U_i + e^{\frac{t_2}{d_2}} \prod_{j=1}^M V_j, \quad (7.45)$$

and the group acting on fields as

$$V_j \rightarrow \gamma_j V_j, \quad j = 1, \dots, M, \quad (7.46)$$

$$U_j \rightarrow \eta_j U_j, \quad j = 1, \dots, M, \quad (7.47)$$

$$U_k \rightarrow \omega_k U_k, \quad k = M + 1, \dots, N, \quad (7.48)$$

where  $\omega_k^{d_1} = 1$ ,  $\gamma_j^{d_2} = 1$ ,  $\gamma_j \eta_j^{d_1} = 1$ , and  $\prod_k \omega_k \prod_j \eta_j = \prod_j \gamma_j = 1$ . Under the condition (7.43), the last two terms of (7.45) are marginal or irrelevant depending on whether the equality holds or not. The two LG orbifolds must be mirror to each other. Indeed both have a central charge  $c/3 = (1 - 1/d_1)(N + M - 2M/d_2)$ .

### 7.3 General Mirror Description

As noted before, for the most general case, what we will find is that the mirror theory can be expressed as an LG theory on a non-compact Calabi-Yau manifold. To see how this works we begin with the case already discussed, i.e. hypersurfaces of degree  $d$  in  $\mathbf{CP}^{N-1}$  and provide an alternative description of the mirror. This reformulation sets the stage for the most general description which follows it.

As noted before, the relevant periods (i.e. D-brane ‘masses’) are given in this case by

$$\Pi = d \int d\Sigma dY_P \prod_{i=1}^N dY_i \Sigma \exp(-\widetilde{W}) \quad (7.49)$$

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<sup>1</sup>Note that this condition is stronger than the non-negativity of the first Chern class.

Since  $d\Sigma$  is given by  $\partial/\partial Y_P$  of the linear terms in  $\widetilde{W}$  we have

$$\begin{aligned}
\Pi &= \int d\Sigma \prod_{i=1}^N dY_i dY_P \frac{\partial}{\partial Y_P} \left[ \exp \left( -\Sigma \left( \sum_{i=1}^N Y_i - dY_P - t \right) \right) \right] \exp \left( -\sum_{i=1}^N e^{-Y_i} - e^{-Y_P} \right) \\
&= \int d\Sigma \prod_{i=1}^N dY_i dY_P e^{-Y_P} \exp \left( -\Sigma \left( \sum_{i=1}^N Y_i - dY_P - t \right) \right) \exp \left( -\sum_{i=1}^N e^{-Y_i} - e^{-Y_P} \right) \\
&= \int \prod_{i=1}^N dY_i dY_P e^{-Y_P} \delta \left( \sum_{i=1}^N Y_i - dY_P - t \right) \exp \left( -\sum_{i=1}^N e^{-Y_i} - e^{-Y_P} \right) \quad (7.50)
\end{aligned}$$

We make the following change of variables

$$e^{-Y_P} = \tilde{P}, \quad (7.51)$$

$$e^{-Y_i} = \tilde{P} U_i, \quad \text{for } i = 1, \dots, d, \quad (7.52)$$

$$e^{-Y_j} = U_j, \quad \text{for } j = d+1, \dots, N. \quad (7.53)$$

Then,

$$\begin{aligned}
\Pi &= \int \prod_{i=1}^N \frac{dU_i}{U_i} d\tilde{P} \delta \left( \log \left( \prod_{i=1}^N U_i \right) + t \right) \exp \left( -\tilde{P} \left( \sum_{i=1}^d U_i + 1 \right) - \sum_{i=d+1}^N U_i \right) \\
&= \int \prod_{i=1}^N \frac{dU_i}{U_i} \delta \left( \log \left( \prod_{i=1}^N U_i \right) + t \right) \delta \left( \sum_{i=1}^d U_i + 1 \right) \exp \left( -\sum_{i=d+1}^N U_i \right). \quad (7.54)
\end{aligned}$$

Thus we have obtained a submanifold  $\widetilde{M}^\circ$  of  $(\mathbf{C}^\times)^N$  defined by

$$\prod_{i=1}^N U_i = e^{-t}, \quad (7.55)$$

$$\sum_{i=1}^d U_i + 1 = 0. \quad (7.56)$$

This is a non-compact manifold of dimension  $N - 2$ . The expression (7.54) is identical to the period of an LG model on  $\widetilde{M}^\circ$  with superpotential

$$W_{\widetilde{M}^\circ} = \sum_{i=d+1}^N U_i. \quad (7.57)$$

This model is the mirror of the sigma model on  $M$ , at least when twisted to topological field theory.

The special case is the case  $d = N$  in which  $M$  is a compact Calabi-Yau manifold. In this case, the superpotential (7.57) is trivial and the mirror is simply the non-linear

sigma model on  $\widetilde{M}^\circ$ . The mirror manifold  $\widetilde{M}^\circ$  is actually an open subset of a Calabi-Yau manifold  $\widetilde{M}$  which is familiar to us. That is, the orbifold of the hypersurface in  $\mathbf{CP}^{N-1}$

$$G(Z_1, \dots, Z_N) = Z_1^N + \dots + Z_N^N + e^{t/N} Z_1 \cdots Z_N = 0, \quad (7.58)$$

by the  $(\mathbf{Z}_N)^{N-2}$  action given by

$$Z_i \mapsto \gamma_i Z_i, \quad \gamma_i^N = 1, \quad \gamma_1 \cdots \gamma_N = 1. \quad (7.59)$$

To see this, we note that

$$U_i = e^{-t/N} \frac{Z_i^N}{Z_1 \cdots Z_N} \quad (7.60)$$

is invariant under the  $\mathbf{C}^\times \times (\mathbf{Z}_N)^{N-2}$  action and solves the first equation (7.55). The second equation (7.56) becomes  $G(Z_i) = 0$ . If  $Z_i$  and  $Z'_i$  yields the same  $U_i$ , it is easy to see that  $Z_i^N = Z'^N_i$  and  $Z_1 \cdots Z_N = Z'_1 \cdots Z'_N$  modulo  $\mathbf{C}^\times$  action. Then, this means  $Z_i = Z'_i$  modulo the  $\mathbf{C}^\times \times (\mathbf{Z}_N)^{N-2}$  action. Thus, the map from  $[Z_i]$  to  $U_i$  is one to one.

Under this identification, we have

$$\begin{aligned} \Pi &= \int \frac{1}{\text{vol}(\mathbf{C}^\times)} \prod_{i=1}^N \frac{dZ_i}{Z_i} \delta \left( \frac{G(Z_1, \dots, Z_N)}{Z_1 \cdots Z_N} \right) \\ &= \int \prod_{i=1}^{N-1} dZ_i \delta(G(Z_1, \dots, Z_{N-1}, 1)) \\ &= \int \left( \prod_{i=1}^{N-2} dZ_i \left/ \frac{\partial G|_{Z_N=1}}{\partial Z_{N-1}} \right|_{G=0} \right) = \int \Omega, \end{aligned} \quad (7.61)$$

which is the period of the holomorphic differential of the Calabi-Yau manifold  $\widetilde{M}$ . Thus, we have shown that the A-twisted sigma model on  $M$  is equivalent to the B-twisted sigma model on  $\widetilde{M}$ .

This tempts us to propose that the sigma model on  $M$  and the LG model on  $\widetilde{M}^\circ$  with the superpotential  $W_{\widetilde{M}^\circ}$  are mirror to each other as (2,2) quantum field theories, not just as topological theories. This is certainly true for  $d = 1$  case with  $M = \mathbf{CP}^{N-2}$  where  $\widetilde{M}^\circ$  is the algebraic torus  $(\mathbf{C}^\times)^{N-2}$  and  $W_{\widetilde{M}^\circ}$  is the affine Toda superpotential. However, for  $d \geq 2$  we must partially compactify  $\widetilde{M}^\circ$  for the model to be the mirror of the sigma model on  $M$ . The reason is that the superpotential  $W_{\widetilde{M}^\circ}$  for  $d > 1$  has a run-away direction which yields a continuous spectrum, a property that we do not expect for a sigma model on a compact smooth manifold  $M$ . The typical case is  $d = N$ ; the sigma model on a non-compact manifold has a continuous spectrum. As we have seen above,  $\widetilde{M}^\circ$  can indeed be compactified to a compact CY manifold  $\widetilde{M}$  and we can claim that the sigma model on  $M$  is mirror to the sigma model on  $\widetilde{M}$ . For  $2 \leq d < N$ , we must compactify only partially

since a non-trivial superpotential is not allowed on a compact complex manifold. In fact, this partial compactification can be found as a simple generalization of the  $d = N$  case. Let us solve the first equation (7.55) for  $U_i$ 's as follows

$$U_i = e^{-t/d} \frac{Z_i^d}{Z_1 \cdots Z_N}, \quad i = 1, \dots, d, \quad (7.62)$$

$$U_j = Z_j^d, \quad j = d+1, \dots, N. \quad (7.63)$$

As in the  $d = N$  case one can see that the map from  $Z_i$  to  $U_i$  is one to one modulo the  $\mathbf{C}^\times \times (\mathbf{Z}_d)^{N-2}$  action given by

$$Z_i \mapsto \lambda \gamma_i Z_i, \quad i = 1, \dots, d, \quad (7.64)$$

$$Z_j \mapsto \gamma_j Z_j, \quad j = d+1, \dots, N, \quad (7.65)$$

where  $\lambda \in \mathbf{C}^\times$  and  $\gamma_i^d = \gamma_j^d = 1$  and  $\gamma_1 \cdots \gamma_N = 1$ . The second equation (7.56) is then expressed as

$$Z_1^d + \cdots + Z_d^d + e^{t/d} Z_1 \cdots Z_d \cdot Z_{d+1} \cdots Z_N = 0. \quad (7.66)$$

This is the equation for a Calabi-Yau hypersurface in  $\mathbf{CP}^{d-1}$  with the  $\psi$  parameter  $\psi = e^{t/d} (Z_{d+1} \cdots Z_N)$ . Thus, the manifold  $\widetilde{M}^\circ$  is partially compactified to a manifold  $\widetilde{M}$  which is the  $(\mathbf{Z}_d)^{N-2}$  quotient of the total space of the family of CY hypersurface in  $\mathbf{CP}^{d-1}$  parametrized by  $\mathbf{C}^{N-d}$  via  $\psi = e^{t/d} (Z_{d+1} \cdots Z_N)$ . Now the superpotential  $W_{\widetilde{M}^\circ}$  (7.57) on  $\widetilde{M}^\circ$  extends to  $\widetilde{M}$  as

$$W_{\widetilde{M}} = Z_{d+1}^d + \cdots + Z_N^d. \quad (7.67)$$

Repeating what we have done in the  $d = N$  case, we can see that the period is expressed as

$$\Pi = \int \Omega_{d-2} \wedge dZ_{d+1} \wedge \cdots \wedge dZ_N \exp(-W_{\widetilde{M}}), \quad (7.68)$$

where  $\Omega_{d-2}$  is the holomorphic  $(d-2)$ -form of the CY hypersurface in  $\mathbf{CP}^{d-1}$ .

Now, the superpotential  $W_{\widetilde{M}}$  on  $\widetilde{M}$  has no run-away direction;  $\widetilde{M}$  includes the limiting points of the run-away direction in  $\widetilde{M}^\circ$ . In particular, we expect that the theory has a discrete spectrum. Thus, we claim that the sigma model on  $M$  is mirror to the LG model on  $\widetilde{M}$  with the superpotential (7.67).

The superpotential (7.67) has  $(N-d)$  non-degenerate critical points at

$$\frac{Z_i^d}{Z_1 \cdots Z_N} = -\frac{e^{t/d}}{d}, \quad i = 1, \dots, d, \quad (7.69)$$

$$Z_j^d = U, \quad j = d+1, \dots, N; \quad U^{N-d} = (-d)^d e^{-t}, \quad (7.70)$$

and a critical manifold at

$$Z_{d+1} = \cdots = Z_N = 0, \quad (7.71)$$

which is the CY hypersurface  $\sum_{i=1}^d Z_i^d = 0$  of  $\mathbf{CP}^{d-1}$ . For  $d > 2$ , this critical CY manifold has dimension  $> 0$  and also the superpotential (7.67) is degenerate there. Thus, we expect that the theory for  $d > 2$  flows in the IR limit to a non-trivial fixed point. This must be equivalent to the non-trivial fixed point studied in section 7.1. For  $d = 2$ , the critical CY manifold is actually a point and the superpotential is non-degenerate. Thus, we expect that the theory has a mass gap for  $d = 2$ . However, because of the orbifolding, the critical point (7.71) may correspond to multiple vacua. From the result of section 7.1.1 the actual number of vacua there is 2 for even  $N$  and 1 for odd  $N$ .

### *Complete Intersection in Toric Variety*

Let  $X$  be the toric variety defined by the charge matrix  $Q_{ia}$  ( $i = 1, \dots, N$ ,  $a = 1, \dots, k$ ). We consider the submanifold  $M$  of  $X$  defined by the equations

$$G_\beta = 0, \quad \beta = 1, \dots, l, \quad (7.72)$$

where  $G_\beta$  are polynomials of  $\Phi_i$  of charge  $d_{\beta a}$  for the  $a$ -th  $U(1)$  gauge group. The sigma model on  $M$  can be realized as the linear sigma model of gauge group  $U(1)^k$  with chiral superfields  $\Phi_i$  of charge  $Q_{ia}$  and  $P_\beta$  of charge  $-d_{\beta a}$  which has a superpotential

$$W = \sum_{\beta=1}^l P_\beta G_\beta(\Phi). \quad (7.73)$$

The theory without this superpotential is the same as the sigma model on a non-compact toric variety  $V$  (defined by charge matrix  $(Q_{ia}, -d_{\beta a})$ ) and has the dual description in terms of the twisted chiral superfield  $\Sigma_a$ ,  $Y_i$  and  $Y_{P_\beta}$  where  $\Sigma_a$  is the field strength of the  $a$ -th gauge group and  $Y_i$  and  $Y_{P_\beta}$  are the dual variables of  $\Phi_i$  and  $P_\beta$ . The dual theory has the twisted superpotential

$$\widetilde{W} = \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^N Q_{ia} Y_i - \sum_{\beta=1}^l d_{\beta a} Y_{P_\beta} - t_a \right) + \sum_{i=1}^N e^{-Y_i} + \sum_{\beta=1}^l e^{-Y_{P_\beta}}. \quad (7.74)$$

As noted before, the period integral for the compact theory is given by

$$\Pi = \int \prod_{a=1}^k d\Sigma_a \prod_{i=1}^N dY_i \prod_{\beta=1}^l dY_{P_\beta} \delta_1 \cdots \delta_l \exp(-\widetilde{W}), \quad (7.75)$$

where

$$\delta_\beta = \sum_{a=1}^k d_{\beta a} \Sigma_a. \quad (7.76)$$

We note that this can be expressed as

$$\delta_\beta = -\frac{\partial}{\partial Y_{P_\beta}} \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^N Q_{ia} Y_i - \sum_{\beta=1}^l d_{\beta a} Y_{P_\beta} - t_a \right). \quad (7.77)$$

Then, via partial integration we obtain

$$\begin{aligned} \Pi &= \int \prod_{a=1}^k d\Sigma_a \prod_{i=1}^N dY_i \prod_{\beta=1}^l dY_{P_\beta} e^{-Y_{P_1}} \dots e^{-Y_{P_l}} \\ &\quad \times \exp \left( - \sum_{a=1}^k \Sigma_a \left( \sum_{i=1}^N Q_{ia} Y_i - \sum_{\beta=1}^l d_{\beta a} Y_{P_\beta} - t_a \right) \right) \exp \left( - \sum_{i=1}^N e^{-Y_i} - \sum_{\beta=1}^l e^{-Y_{P_\beta}} \right) \\ &= \int \prod_{i=1}^N dY_i \prod_{\beta=1}^l e^{-Y_{P_\beta}} dY_{P_\beta} \prod_{a=1}^k \delta \left( \sum_{i=1}^N Q_{ia} Y_i - \sum_{\beta=1}^l d_{\beta a} Y_{P_\beta} - t_a \right) \\ &\quad \times \exp \left( - \sum_{i=1}^N e^{-Y_i} - \sum_{\beta=1}^l e^{-Y_{P_\beta}} \right). \end{aligned} \quad (7.78)$$

This is the expression for the BPS mass for the most general toric complete intersection. Now let us consider the case where one can find  $n_\beta^i = 0$  or 1 such that

$$\sum_{i=1}^N n_\beta^i Q_{ia} = d_{\beta a}. \quad (7.79)$$

If we make the following change of variables

$$e^{-Y_{P_\beta}} = \tilde{P}_\beta, \quad (7.80)$$

$$e^{-Y_i} = U_i \prod_{\beta=1}^l \tilde{P}_\beta^{n_\beta^i}, \quad (7.81)$$

the period integral is expressed as

$$\begin{aligned} \Pi &= \int \prod_{i=1}^N \frac{dU_i}{U_i} \prod_{\beta=1}^l d\tilde{P}_\beta \prod_{a=1}^k \delta \left( \log \left( \prod_{i=1}^N U_i^{Q_{ia}} \right) + t_a \right) \exp \left( - \sum_{\beta=1}^l \tilde{P}_\beta \left( \sum_{n_\beta^i=1} U_i + 1 \right) - \sum_{n_\beta^j=0, \forall \beta} U_j \right) \\ &= \int \prod_{i=1}^N \frac{dU_i}{U_i} \prod_{a=1}^k \delta \left( \log \left( \prod_{i=1}^N U_i^{Q_{ia}} \right) + t_a \right) \prod_{\beta=1}^l \delta \left( \sum_{n_\beta^i=1} U_i + 1 \right) \times \exp \left( - \sum_{n_\beta^j=0, \forall \beta} U_j \right) \end{aligned} \quad (7.82)$$

Thus, we have obtained a submanifold  $\tilde{M}^\circ$  of  $(\mathbf{C}^\times)^N$  defined by

$$\prod_{i=1}^N U_i^{Q_{ia}} = e^{-t_a}, \quad (7.83)$$

$$\sum_{n_\beta^i=1} U_i + 1 = 0, \quad (7.84)$$

where the sum in (7.84) is over  $i$  such that  $n_\beta^i = 1$ . This is a non-compact manifold of dimension  $(N - k - l)$  which is the same as the dimension of  $M$ . The period (7.82) is identical to the period of an LG model on  $\widetilde{M}^\circ$  with superpotential

$$W_{\widetilde{M}^\circ} = \sum_{n_\beta^j=0, \forall \beta} U_j, \quad (7.85)$$

where the sum here is over  $j$  such that  $n_\beta^j = 0$  for all  $\beta$ . Thus, we have shown that the mirror of the sigma model on  $M$  is given by this LG model on  $\widetilde{M}^\circ$ , at least when twisted to topological field theory. The expression (7.82) of the period on the mirror manifold (7.83)-(7.84) with the superpotential (7.85) is equivalent to the one that appears in [16].

## 8 Directions for Future Work

In this paper we have seen how mirror symmetry, formulated as a duality between pairs of 2d QFT's can be proven using rather simple physical ideas. Not only have we recovered the known formulation for mirror symmetry, but we have also generalized it to classes which were not known before. In particular, we have shown that a theory and the mirror will give rise to the same “BPS mass” for the D-branes.

It is natural to ask to what extent one can derive dualities between QFT's in higher dimensions. One idea along this line is their reduction to 2 dimensions. For example, it was shown in [98] how certain results about 4 dimensional  $N = 2$  Yang-Mills theory compactified on a Riemann surface to 2 dimensions can be related to sigma models, and computed via mirror symmetry. Given that we have now understood how mirror symmetry can be derived, we can go back and ask to what extent all the quantum corrections of  $N = 2$  theories can be understood using results about 2d QFT's. It would not be surprising if all the BPS data [38], which is all we know at the present about the  $N = 2$  theories in 4 dimensions, can be derived in this way. This in fact fits with the idea that in supersymmetric theories certain exact results can be computed rather easily. For example computation of Witten's index can be reduced to an ordinary integral. It is thus conceivable that the computations related to exact BPS aspects for  $N = 2$  theories in 4 dimensions can be related to derivable facts about 2 dimensional quantum field theories. This program, we believe, is an interesting one to investigate.

At least in one case it already seems to work: Consider pure  $N = 1$  supersymmetric Yang-Mills theory in 4 dimensions. It is known that upon compactification to 3 dimensions one obtains an effective theory captured by a superpotential of affine Toda type [39, 99–101]. The same superpotential survives in the compactification down to 2 dimensions.

This can now be derived in our setup as follows: Consider compactification of the four dimensional theory on  $T^2$ , down to two dimensions. The effective theory in two dimensions will be a sigma model whose target space is the moduli space of flat connection of the corresponding group on  $T^2$ . These spaces have been studied [102] and is found to be the weighted projective space  $\mathbf{CWP}_{(1,g_1,\dots,g_r)}^r$  where  $r$  is the rank of the gauge group and  $g_1, \dots, g_r$  are its coroot integers. The  $(2,2)$  sigma model on this space is realized as a  $U(1)$  gauge theory with chiral matter fields of charge  $(1, g_1, \dots, g_r)$ . Using the mirror symmetry results of this paper, we obtain the mirror affine Toda potential

$$\widetilde{W} = \Lambda \left( e^{\Theta_1} + \dots + e^{\Theta_r} + e^{-\sum_{i=1}^r g_i \Theta_i} \right), \quad (8.1)$$

in agreement with [39, 99–101]. It would be interesting to extend these results to other  $N = 1$  theories in four dimensions<sup>1</sup>.

Another aspect of our work which may find further applications is the issue of the meeting of Higgs and Coulomb branch in 2 dimensional gauge theories. This issue has been studied in [104]. The dual of gauge theory coupled to matter that we have found in this paper seems potentially useful for studying such questions. In particular the dual of the matter fields (whose vevs mark the Higgs branch) and the scalar in the vector multiplet (whose vev marks the Coulomb branch) are in the same type of multiplet and the superpotential we have found captures certain aspects of the gauge dynamics involving these fields. Another application of our work may be to the large  $N$  behavior of Chern-Simons theory on  $S^3$ , which has been conjectured to be dual to topological string on  $\mathcal{O}(-1) + \mathcal{O}(-1)$  over  $\mathbf{CP}^1$  [105].

In appendix A we have presented a conjectured generalization of the duality in  $(2,2)$  supersymmetric gauge theories in 2 dimensions for non-abelian groups, motivated by considering a generic point on the Coulomb branch of these theories. It would be interesting to check the validity of this conjecture. This conjecture leads to the computation of BPS structure for complete intersections defined on arbitrary flag varieties.

Certain aspects of our work may also be relevant for the coupling of topological sigma model to topological gravity. In fact, it is tempting to conjecture that some matrix version of the toda theories we have found would be relevant for that reformulation. This would be interesting to study further.

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<sup>1</sup>For another relation between moduli of flat connections on  $T^2$  and mirror symmetry, see [103].



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## Appendix

### A Non-Abelian Gauge Theories and Complete Intersections in Grassmannians— A Conjecture

In this paper we have been mainly dealing with linear sigma models realized as complete intersection in toric varieties. We have used abelian gauge theories in analyzing them. It is natural to ask if we can compute analogous quantities for the case of complete intersections in Grassmannians. In fact this class can also be realized as gauge theories as well [10], but in this case it will involve a  $U(N)$  gauge theory coupled to matter. So we would need to know the analog of the dual formulation for this gauge system.

Our argument for abelian gauge theory is not applicable to non-abelian gauge theories. Here we attempt to make a conjecture for what the analog dual is, which we motivate using the results already obtained for the product of  $U(1)$ 's. We also make some checks for the validity of the conjecture.

We consider the  $U(N)$  gauge theory coupled to matter in a (not necessarily irreducible) representation  $\mathcal{R}$ .<sup>1</sup> The  $U(N)$  gauge supermultiplet contains a complex scalar in the adjoint representation of  $U(N)$ . The “generic”<sup>2</sup> configuration which survives in the infrared is the one corresponding to the complex scalar given by a diagonal matrix. This is the generic point on the Coulomb branch of the theory. Let  $\Sigma_i$  denote the fields corresponding to these diagonal elements. These are well defined up to permutation. In other words the invariant fields involve symmetric polynomials in  $\Sigma_i$ . In the generic configuration for  $\Sigma_i$  the theory becomes a  $U(1)^N/S_N$  gauge theory. So we could then apply our results for the product of  $U(1)$  gauge theories to obtain the dual, modulo taking into account the permutation action on the groups. There will be  $\dim \mathcal{R}$  fields  $Y^\alpha$  obtained by dualizing the matter field in representation  $R$ . Each  $Y_\alpha$  corresponds to a weight of  $U(N)$  lattice, and we can associate the corresponding  $U(1)^N$  charges  $Q_i^\alpha$  to them. Consider the superpotential

$$W = \sum_i \Sigma_i (Q_i^\alpha Y^\alpha - t) + \sum_\alpha e^{-Y_\alpha} \quad (\text{A.1})$$

where  $t$  denotes the FI parameter for the  $U(N)$  gauge theory. This is almost what we propose as the dual, except that we now have to consider Weyl invariant (i.e.  $S_N$  invariant) combination of fields. The  $S_N$  acts on  $\Sigma_i$  by permutation, as already noted. It acts on  $Y^\alpha$  by the permutation induced on the weights of the representation  $\mathcal{R}$  by the action of

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<sup>1</sup>One can generalize this conjecture to arbitrary groups

<sup>2</sup>This is why we cannot prove our conjecture: The places where some  $\Sigma_i$  and  $\Sigma_j$  are equal yields a non-abelian unbroken group and we are assuming that this does not cause any problems for our dualization.

the Weyl group. Clearly the above action is invariant under the Weyl group action. We then consider the theory given by

$$W_{invariant} = W//S_N \quad (\text{A.2})$$

where by this we mean that the fundamental fields of the theory are to be written in terms of the  $S_N$  invariant combinations (note that this is not the same as orbifolding the theory).

Concretely what this would mean in computing the periods (i.e. D-brane masses) is as follows: We will consider integrals of the form

$$\int \prod_i d\Sigma_i \prod_\alpha dY^\alpha \prod_{i<j} (\Sigma_i - \Sigma_j) e^{-W} \quad (\text{A.3})$$

The insertion of  $\Delta = \prod_{i<j} (\Sigma_i - \Sigma_j)$  is to make the measure correspond to the symmetric measure, and is reminiscent of the  $\delta$  insertion discussed in the context of getting hypersurfaces from non-compact toric varieties. Note that this also agrees with the dimension count for the space. Our proposed dual has infrared degrees of freedom which is too big:  $\dim \mathcal{R} - N$  whereas it should have had  $\dim \mathcal{R} - N^2$ . Insertion of  $\Delta^2$  in the correlation functions (which is equivalent to the insertion of  $\Delta$  in the periods) changes the dimension count by  $N(N-1)$ , and makes up for the discrepancy. This is very similar to how the insertion of  $\delta$ 's in the non-compact toric case was used to embed the compact cohomology ring in the non-compact one.

The above periods can be easily computed from the corresponding non-compact toric varieties: The insertion of  $\Delta$  is the same as the action of  $\prod_{i<j} (\partial/\partial t_i - \partial/\partial t_j)$  acting on the periods of the corresponding non-compact toric case (and substituting  $t_i = t_j = t$  at the end). Similarly the case of hypersurfaces in Grassmannians would be obtained from the non-compact version of bundles over Grassmannians by inclusions of extra insertions similar to  $\delta$ . Also this conjecture naturally extends to the case of flag manifolds and complete intersections in them.

The above conjecture would be interesting to verify. For the case of Grassmannian itself, this conjecture gives the correct ring structure [26], which was one of the motivations for the above conjecture. It would also be interesting to connect the above formulation with the corresponding LG model proposed for the Grassmannian [28] which was further generalized in [96, 97] for other homogeneous spaces and complete intersections therein.

## B Supersymmetry transformation

We record here the supersymmetry transformation of the vector and chiral multiplet fields (in the Wess-Zumino gauge) [10, 47]. This is obtained by dimensional reduction of the formulae in [46] for  $N = 1$  supersymmetry transformation in  $3 + 1$  dimensions. The reduction is in  $x^1, x^2$  directions and the scalars in the vector multiplet is defined as  $\sigma = (v_1 - iv_2)$  and  $\bar{\sigma} = (v_1 + iv_2)$ . The time coordinate is still  $x^0$  but we rename the spacial coordinate  $x^3$  as  $x^1$ . (The normalization of vector multiplet fields used in this paper differs from the one in [10, 47] by factors of  $\sqrt{2}$ : if we denote the latter as  $\Sigma', \sigma',$  etc, the relations are  $\Sigma = \sqrt{2}\Sigma', \sigma = \sqrt{2}\sigma', \lambda_{\pm} = \sqrt{2}\lambda'_{\pm}, D = D'$  and  $F_{01} = F'_{01}$ .)

The four supersymmetry generators are combined as

$$\delta = \epsilon^{\alpha} Q_{\alpha} + \bar{\epsilon}_{\alpha} \bar{Q}^{\alpha} = \epsilon_{+} Q_{-} - \epsilon_{-} Q_{+} - \bar{\epsilon}_{+} \bar{Q}_{-} + \bar{\epsilon}_{-} \bar{Q}_{+}, \quad (\text{B.1})$$

where  $\epsilon^{\pm}$  and  $\bar{\epsilon}^{\pm}$  are anti-commuting spinorial parameters ( $\epsilon^{\mp} = \pm \epsilon_{\pm}$  etc). The transformation of the vector multiplet fields is

$$\begin{aligned} \delta v_{\pm} &= \sqrt{2}i\bar{\epsilon}_{\pm}\lambda_{\pm} + \sqrt{2}i\epsilon_{\pm}\bar{\lambda}_{\pm}, \\ \delta\sigma &= -\sqrt{2}i\bar{\epsilon}_{+}\lambda_{-} - \sqrt{2}i\epsilon_{-}\bar{\lambda}_{+}, \\ \delta\bar{\sigma} &= -\sqrt{2}i\epsilon_{+}\bar{\lambda}_{-} - \sqrt{2}i\bar{\epsilon}_{-}\lambda_{+}, \\ \delta D &= \frac{1}{\sqrt{2}}\left(-\bar{\epsilon}_{+}D_{-}\lambda_{+} - \bar{\epsilon}_{-}D_{+}\lambda_{-} + \epsilon_{+}D_{-}\bar{\lambda}_{+} + \epsilon_{-}D_{+}\bar{\lambda}_{-} \right. \\ &\quad \left. + \epsilon_{+}[\sigma, \bar{\lambda}_{-}] + \epsilon_{-}[\bar{\sigma}, \bar{\lambda}_{+}] - \bar{\epsilon}_{-}[\sigma, \lambda_{+}] - \bar{\epsilon}_{+}[\bar{\sigma}, \lambda_{-}]\right), \\ \delta\lambda_{+} &= \sqrt{2}i\epsilon_{+}(D + iF_{01} + \frac{i}{2}[\sigma, \bar{\sigma}]) + \sqrt{2}\epsilon_{-}D_{+}\bar{\sigma}, \\ \delta\lambda_{-} &= \sqrt{2}i\epsilon_{-}(D - iF_{01} - \frac{i}{2}[\sigma, \bar{\sigma}]) + \sqrt{2}\epsilon_{+}D_{-}\sigma, \\ \delta\bar{\lambda}_{+} &= -\sqrt{2}i\bar{\epsilon}_{+}(D - iF_{01} + \frac{i}{2}[\sigma, \bar{\sigma}]) + \sqrt{2}\bar{\epsilon}_{-}D_{+}\sigma, \\ \delta\bar{\lambda}_{-} &= -\sqrt{2}i\bar{\epsilon}_{-}(D + iF_{01} - \frac{i}{2}[\sigma, \bar{\sigma}]) + \sqrt{2}\bar{\epsilon}_{+}D_{-}\bar{\sigma}, \end{aligned} \quad (\text{B.2})$$

where  $v_{\pm} = v_0 \pm v_1$ ,  $D_{\pm} = D_0 \pm D_1$ . The transformation of the charged chiral multiplet fields is given by

$$\begin{aligned} \delta\phi &= \sqrt{2}\epsilon_{+}\psi_{-} - \sqrt{2}\epsilon_{-}\psi_{+}, \\ \delta\psi_{+} &= \sqrt{2}i\bar{\epsilon}_{-}D_{+}\phi + \sqrt{2}\epsilon_{+}F - \sqrt{2}\bar{\epsilon}_{+}\bar{\sigma}\phi, \\ \delta\psi_{-} &= -\sqrt{2}i\bar{\epsilon}_{+}D_{-}\phi + \sqrt{2}\epsilon_{-}F + \sqrt{2}\bar{\epsilon}_{-}\sigma\phi, \\ \delta F &= -\sqrt{2}i\bar{\epsilon}_{+}D_{-}\psi_{+} - \sqrt{2}i\bar{\epsilon}_{-}D_{+}\psi_{-} \\ &\quad + \sqrt{2}(\bar{\epsilon}_{+}\bar{\sigma}\psi_{-} + \bar{\epsilon}_{-}\sigma\psi_{+}) + \sqrt{2}i(\bar{\epsilon}_{-}\bar{\lambda}_{+} - \bar{\epsilon}_{+}\bar{\lambda}_{-})\phi. \end{aligned} \quad (\text{B.3})$$

From (B.2) it is clear that turning on an expectation value of the scalar component of the vector multiplet  $\sigma = \widetilde{m}$  and freezing the fluctuation of the entire vector multiplet (by setting the coupling zero) does not break any supersymmetry provided  $\widetilde{m}$  and its hermitian conjugate  $\overline{\widetilde{m}}$  commute with each other

$$[\widetilde{m}, \overline{\widetilde{m}}] = 0. \quad (\text{B.4})$$

Thus, twisted mass for an abelian subgroup of the flavor symmetry preserves  $(2, 2)$  supersymmetry. The same thing can be said also for holomorphic isometry of a Kahler manifold (see the formulae (28)-(30) in [49]).

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